Mechanics & Materials 1

Chapter 15
Stress and Strain
Transformation

FAMU-FSU College of Engineering
Department of Mechanical Engineering
General State of Stress

- In general, the three dimensional state of stress at a point in a body can be represented by nine components:
  - $\sigma_{xx}$, $\sigma_{yy}$, and $\sigma_{zz}$: Normal stresses
  - $\tau_{xy}$, $\tau_{yx}$, $\tau_{xz}$, $\tau_{yz}$, $\tau_{zx}$, and $\tau_{zy}$: Shear stresses

- By equilibrium, we can show that there are only six independent components of the stress $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{zz}$, $\tau_{xy}$, $\tau_{xz}$, and $\tau_{yz}$
Plane Stress

• In much of engineering stress analysis, the condition of **plane stress** applies.

• **Plane Stress**: one of the three normal stresses, usually $\sigma_z$ vanishes and the other two normal stresses $\sigma_x$ and $\sigma_y$, and the shear stress $\tau_{xy}$ are known.
Plane Stress Transformation: Finding Stresses on Various Planes

• General Problem:

• * Given two coordinate systems, x- y and x' - y', and a stress state defined relative to the first coordinate system \( xyz \) : \( \sigma_x, \sigma_y, \tau_{xy} \)

• * Find the stress components relative to the second coordinate system \( x'y'z' \) : \( \sigma'_x, \sigma'_y, \tau'_{xy} \)
Consider a triangular block of uniform thickness, $t$:

For equilibrium:

$$
\sum F_{x'} = tL\sigma'_{x} \\
= -t(L\cos\theta)\sigma_{x}\cos\theta - t(L\cos\theta)\tau_{xy}\sin\theta \\
- t(L\sin\theta)\sigma_{y}\sin\theta - t(L\sin\theta)\tau_{xy}\cos\theta
$$

$$
\sum F_{y'} = tL\tau'_{xy} \\
= +t(L\cos\theta)\sigma_{x}\sin\theta - t(L\cos\theta)\tau_{xy}\cos\theta \\
- t(L\sin\theta)\sigma_{y}\cos\theta - t(L\sin\theta)\tau_{xy}\sin\theta = 0
$$
• Simplifying:

\[
\sigma'_x = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2 \tau_{xy} \sin \theta \cos \theta
\]

\[
\tau'_{xy} = \tau_{xy} (\cos^2 \theta - \sin^2 \theta) - (\sigma_x - \sigma_y) \sin \theta \cos \theta
\]

• Using:

\[
\cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta) \quad \sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta)
\]

\[
\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta
\]
Transformation Equations for Plane Stress

\[ \sigma'_{x} = \frac{\sigma_{x} + \sigma_{y}}{2} + \frac{\sigma_{x} - \sigma_{y}}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \]

\[ \sigma'_{y} = \frac{\sigma_{x} + \sigma_{y}}{2} - \frac{\sigma_{x} - \sigma_{y}}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \]

\[ \tau'_{xy} = -\frac{\sigma_{x} - \sigma_{y}}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \]

\[ \Rightarrow \sigma'_{x} + \sigma'_{y} = \sigma_{x} + \sigma_{y} \]
Special Cases of Plane Stress

1. Uniaxial Stress State: \( \sigma_y = \tau_{xy} = 0 \)

\[
\begin{align*}
\sigma'_x &= \frac{1}{2} \sigma_x (1 + \cos 2\theta) = \sigma_x \cos^2 \theta \\
\sigma'_y &= \frac{1}{2} \sigma_x (1 - \cos 2\theta) = \sigma_x \sin^2 \theta \\
\tau'_{xy} &= -\frac{1}{2} \sigma_x \sin 2\theta = -\sigma_x \sin \theta \cos \theta
\end{align*}
\]

2. Biaxial Stress State: \( \tau_{xy} = 0 \)

\[
\begin{align*}
\sigma'_x &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta \\
\sigma'_y &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta \\
\tau'_{xy} &= -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta
\end{align*}
\]
3. Pure Shear: $\sigma_x = \sigma_y = 0$

$$\sigma'_x = \tau_{xy} \sin 2\theta = 2\tau_{xy} \sin \theta \cos \theta$$

$$\sigma'_y = -\tau_{xy} \sin 2\theta = -2\tau_{xy} \sin \theta \cos \theta$$

$$\tau'_{xy} = \tau_{xy} \cos 2\theta = \tau_{xy} (\cos^2 \theta - \sin^2 \theta)$$
Principal Stress

• $\sigma'_x$ varies as a function of the angle $\theta$

$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

• The maximum and minimum values of $\sigma'_x$ are called the principal stresses. To find the max and min values: \[
\frac{d\sigma'_x}{d\theta} = 0
\]

$$\frac{d\sigma'_x}{d\theta} = -(\sigma_x - \sigma_y) \sin 2\theta + 2\tau_{xy} \cos 2\theta = 0$$

$$\Rightarrow \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

• Where $\theta_p$ defines the orientation of the principle planes on which the principle stress act.
• Two values of $2\theta_p$ in the range of 0 to 360 satisfy this equation.

• These two values differ by 180° so that $\theta_p$ has two values that differ by 90°, one between 0 and 90° and the other between 90° and 180°.

• For one of the angles $\theta_p$, the stress is a maximum principal stress ($\sigma_1$) and for the other it is a minimum ($\sigma_2$).

• Because the two values of $\theta_p$ are 90° apart, => the principal stress occurs on mutually perpendicular planes.
Principal Stress

To calculate $\theta_p$, consider the triangle

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2R}$$

$$\sin 2\theta_p = \frac{\tau_{xy}}{R}$$

Subbing back in yields the principal stresses:

- OR

$$\rightarrow \sigma_{1,2} = \sigma_{\text{avg}} \pm R$$

where

$$\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2}$$
The shear stress corresponding to the principal stress direction is given by:

\[ \tau_{12} = \tau_{xy}(\theta_p) \]

\[ = -\left( \frac{\sigma_x - \sigma_y}{2} \right) \frac{\tau_{xy}}{R} + \tau_{xy} \left( \frac{\sigma_x - \sigma_y}{2R} \right) \]

\[ = 0 \]

The shear stress is identically zero in the principal stress directions! (biaxial stress state)
Maximum Shear Stress

- To find maximum shear:
  \[
  \frac{d\tau'_{xy}}{d\theta} = 0
  \]

  \[
  \frac{d\tau^l_{xy}}{dx} = -(\sigma_x - \sigma_y)\cos2\theta - 2\tau_{xy}\sin2\theta = 0 \quad \Rightarrow \quad \tan2\theta_s = -\left(\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right)
  \]

- Where \(\theta_s\) defines the angle of the planes of maximum shear stress.

![Diagram](image-url)
• From trigonometry
\[ \theta_s = \theta_p \pm 45^\circ \]

• The planes of maximum shear stress occur at 45° to the principal planes.

• Subbing back in yields max shear:
\[
\tau_{\text{max}} = \frac{\sigma_x - \sigma_y}{2} + \tau_{xy} = R
\]

\[
\tau_{\text{max}} = \tau_{xy}(\theta_s)
\]

\[
\frac{\sigma_1 - \sigma_2}{2}
\]
Maximum Shear Stress

- The normal stress corresponding to the max shear stress direction is given by:

\[
\sigma_x' = \sigma_x'^{\left(\theta_s\right)} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \frac{\tau_{xy}}{R} - \tau_{xy} \left(\frac{\sigma_x - \sigma_y}{2R}\right) = \sigma_{avg}
\]
Summary of Equations

\[
\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \theta_s = \theta_p \pm 45^\circ
\]

\[
R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}
\]

\[
\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}
\]

\[
\sigma_1 = \sigma_{avg} + R
\]

\[
\sigma_2 = \sigma_{avg} - R
\]

\[
\tau_{\text{max}} = R = \frac{\sigma_1 - \sigma_2}{2}
\]
Mohr's Circle for Plane Stress

- Recall the plane stress transformation equations:

\[
\sigma'_x = \sigma_{avg} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta
\]

\[
\tau'_{xy} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta
\]

- Rearrange to get

\[
\left(\sigma'_x - \sigma_{avg}\right)^2 + \left(\tau'_{xy}\right)^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2
\]

\[
\Rightarrow \left(\sigma'_x - \sigma_{avg}\right)^2 + \left(\tau'_{xy}\right)^2 = R^2
\]

- The above equation is for a circle of radius R and Center $\sigma_{avg}$
Mohr’s Circle for Plane Stress

- Mohr’s circle equation

\[
\left( \sigma'_x - \sigma_{\text{avg}} \right)^2 + \left( \tau'_xy \right)^2 = R^2
\]

- Equation of a circle in the $\sigma'_x, \tau'_xy$ plane centered at $|(\sigma_{\text{avg}}, 0)|$ and radius $R$

* Every plane becomes a point on the circle.
* The intersection with the $\sigma'_x$ axis defines the principal stresses.

\[
\lambda + \frac{\sigma_{\text{avg}}}{\gamma_{\text{min}}} \sigma = \sigma
\]

\[
\lambda - \frac{\sigma_{\text{avg}}}{\gamma_{\text{min}}} \sigma = \sigma
\]

* The bottom and top center positions correspond to:

\[
\tau_{\text{max}} = R \quad \text{and} \quad \tau_{\text{min}} = -R
\]
Procedures to Construct Mohr’s Circle

With $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ known, the procedure for constructing Mohr's circle is as follows;

1) Draw a set of coordinate axes with $\sigma$ as abscissa (positive to the right) and $\tau$ as ordinate (positive upward)

2) Locate the center C of the circle at the point having coordinates $(\sigma_{\text{aver}}, 0)$

3) Locate point A, representing the stress conditions on the face A $(\sigma_x, -\tau_{xy})$

4) Locate point B, representing the stress conditions on the face B $(\sigma_y, \tau_{xy})$

5) Draw a line from point A to point B. This line is a diameter of the circle and passes through the center C

6) Using point C as the center, draw Mohr's circle through points A and B.

7) On the circle, we measure an angle $2\theta$ clockwise from radius CA. The angle $2\theta$ locates point D.

Point D on the circle represents the stresses on the face D of the element.

Note that an angle $2\theta$ on Mohr's circle corresponds to an angle $\theta$ on a stress element.
Procedures to Construct Mohr’s Circle
Plane Strain

- Plane strain is defined by the strain state \((\varepsilon_x, \varepsilon_y, \gamma_{xy})\); it is the limiting condition in the center plane of a very thick specimen.

- Consider a rectangular element of material, OABC, in the xy-plane shown in Figure; it is required to find the normal and shearing strains in the direction of the diagonal OB, when the normal and shearing strains in the directions Ox, Oy are given.
Strain Transformation

- Assume that strain transformation is desired from an $xy$ coordinate system to an $xy'$ set of axes, where the latter is rotated counterclockwise ($+\theta$) from the $xy$ system.

- The transformation equations for plane strain are

$$\varepsilon_x' = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\varepsilon_y' = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta - \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta$$

Element of size $dx$ by $dy$ at angle $\theta$ before and after the application of biaxial stresses, showing its deformation.
Principal Strains

• For isotropic materials only, principal strains (with no shear strain) occur along the principal axes for stress.

• In plane strain the principal strains $\varepsilon_1$ and $\varepsilon_2$ are expressed as

$$\varepsilon_{12} = \frac{\varepsilon_x + \varepsilon_y}{2} + \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

• The angular position $\theta_p$ of the principal axes (measured positive counterclockwise) with respect to the given $xy$ system is determined from

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}.$$
Maximum Shear Strain

- Like in the case of stress, the maximum in-plane shear strain is:
  \[ \frac{\gamma_{x'y'}_{\text{max}}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \]

- The corresponding average normal strain is:
  \[ \tan 2\theta = \frac{\varepsilon_x - \varepsilon_y}{\gamma_{xy}} \]
  \[ \varepsilon_{\text{avg}} = \frac{\varepsilon_x + \varepsilon_y}{2} \]
Mohr’s Circle of Strain

- The direct and shearing strains in an inclined direction are given by relations which are similar to the Equations for the direct and shearing stresses on an inclined plane.

- This suggests that the strains in any direction can be represented graphically in a similar way to the stress system.

\[
\begin{align*}
\epsilon_x' &= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \\
\epsilon_y' &= \frac{\epsilon_x + \epsilon_y}{2} - \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta - \frac{\gamma_{xy}}{2} \sin 2\theta \\
\gamma_{xy}' &= -\frac{\epsilon_x - \epsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta
\end{align*}
\]
• As in the case of stress, there is a graphical overview by Mohr’s circle of the directional dependence of the normal and shear strain components at a point in a material. This circle has a center $C$ at $\varepsilon_{ave} = (\varepsilon_x + \varepsilon_y)/2$ which is always on the $\varepsilon$ axis, but is shifting left and right in a dynamic loading situation. The radius $R$ of the circle is

$$R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$
Mohr’s Circle of Strain

- For given values of $\varepsilon_x$, $\varepsilon_y$ and $\gamma_{xy}$ it is constructed in the following way:
- Two mutually perpendicular axes, $\varepsilon$ and $\gamma/2$, are set up
- The points $(\varepsilon_x, \gamma_{xy}/2)$ and $(\varepsilon_y, -\gamma_{xy}/2)$ are located; the line joining these points is a diameter of the circle of strain.
- The values of $\varepsilon$ and $\gamma/2$ in an inclined direction making an angle $\theta$ with Ox are given by the points on the circle at the ends of a diameter making an angle $2\theta$ with PQ; the angle $2\theta$ is measured clockwise.

Mohr’s circle of strain: the diagram is similar to the circle of stress, except that $\gamma/2$ is plotted along the ordinates and not $\gamma$. 
We note that the maximum and minimum values of $\varepsilon$, given by $\varepsilon_1$ and $\varepsilon_2$ occur when $\gamma/2$ is zero; $\varepsilon_1$, $\varepsilon_2$ are called principal strains, and occur for directions in which there is no shearing strain.

Mohr’s Circle of Strain
• Define the terms $\varepsilon_{xx}$, $\varepsilon_{yy}$, $\gamma_{xy}$ as the strains of an element of size $(dx \times dy)$ at an angle $\theta$ with respect to the horizontal axis.

• Then the equations which defines these strains are:

$$\varepsilon_{xx} = \varepsilon_{xx} \cos^2 \theta + \varepsilon_{yy} \sin^2 \theta + \gamma_{xy} \cos \theta \sin \theta$$

• If the strain at any angle could be measured, the equation above can then be used to determine the direct and shear strains in the structure about the x & y axes.

• These measurements are done using a Strain Gauge Rosette.
A normal arrangement is to have three strain gauges oriented at three different angles w.r.t. the horizontal axis of the structure, like this:

Because we have three unknowns terms and you want to find, $\varepsilon_{xx}$, $\varepsilon_{yy}$, $\gamma_{xy}$, use equation

$$\varepsilon_{\theta} = \varepsilon_{x} \cos^2 \theta + \varepsilon_{y} \sin^2 \theta + \gamma_{xy} \sin \theta \cos \theta$$

three times, once for each angle. Then solve for the three strain

$$\varepsilon_{\theta_a} = \varepsilon_{x} \cos^2 \theta_a + \varepsilon_{y} \sin^2 \theta_a + \gamma_{xy} \sin \theta_a \cos \theta_a$$
$$\varepsilon_{\theta_b} = \varepsilon_{x} \cos^2 \theta_b + \varepsilon_{y} \sin^2 \theta_b + \gamma_{xy} \sin \theta_b \cos \theta_b$$
$$\varepsilon_{\theta_c} = \varepsilon_{x} \cos^2 \theta_c + \varepsilon_{y} \sin^2 \theta_c + \gamma_{xy} \sin \theta_c \cos \theta_c$$
Strain rosettes are often arranged in 45 or 60 patterns, such that the solutions for the unknowns will be as follows:

For the $45^0$ Rosette: $(\theta_a = 0^0, \theta_b = 45^0, \theta_c = 90^0)$

\[
\begin{align*}
\varepsilon_x &= \varepsilon_a \\
\varepsilon_y &= \varepsilon_c \\
\gamma_{xy} &= 2\varepsilon_b - (\varepsilon_a + \varepsilon_c)
\end{align*}
\]

For the $60^0$ Rosette: $(\theta_a = 0^0, \theta_b = 60^0, \theta_c = 120^0)$

\[
\begin{align*}
\varepsilon_x &= \varepsilon_a \\
\varepsilon_y &= \frac{1}{3} (2\varepsilon_b + 2\varepsilon_c - \varepsilon_a) \\
\gamma_{xy} &= \frac{2}{\sqrt{3}} (\varepsilon_b - \varepsilon_c)
\end{align*}
\]
Material - Property Relationships

- **Generalized Hooks Law**: Once we have the strains use the relationships between stress and strain to find the stresses: (for isotropic material)

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E} \left( \sigma_x - \nu (\sigma_y + \sigma_z) \right) \\
\varepsilon_y &= \frac{1}{E} \left( \sigma_y - \nu (\sigma_x + \sigma_z) \right) \\
\varepsilon_z &= \frac{1}{E} \left( \sigma_z - \nu (\sigma_y + \sigma_x) \right) \\
\gamma_{xy} &= \frac{\tau_{xy}}{G} \\
\gamma_{xz} &= \frac{\tau_{xz}}{G} \\
\gamma_{yz} &= \frac{\tau_{yz}}{G}
\end{align*}
\]

Where

- **E**: Young’s modulus,
- **\( \nu \)**: Poisson’s ratio
- **G**: Shear modulus

\[
G = \frac{E}{2(1+\nu)}
\]
Dilatation

• Under the application of normal stresses, the volume of the material will change.

• The change in volume per unit volume (dV/V) is called the dilatation: $e$

\[
e = \frac{dV}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z
\]
A material under the action of equal compressive stresses \( s \) in three mutually perpendicular directions, is subjected to a **hydrostatic pressure**, \( p \). The term hydrostatic is used because the material is subjected to the same stresses as would occur if it were immersed in a fluid at a considerable depth.

The ratio between the hydrostatic pressure and the dilatation is called the **Bulk modulus** : \( k \)

\[
k = \frac{p}{e} = \frac{E}{3(1 - 2\nu)}
\]