2.4 | Navier-Stokes

weak form

\[ \iint \left[ w_1 \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + w_1 \left( \frac{1}{\rho} \frac{\partial p}{\partial x} - \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right) \right] \, dx \, dy = 0 \]

\[ \iint \left[ w_2 \left( \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + w_2 \left( \frac{1}{\rho} \frac{\partial p}{\partial y} - \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right) \right] \, dx \, dy = 0 \]

\[ \iint w_3 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \, dx \, dy = 0 \leq \text{final weak form} \]

the terms \( w_1 \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) \) and \( w_2 \left( \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) \) are not changed.

The last two are because they are used in the boundary conditions. See (3) in problem (pg. 99).

\[ \iint \left[ w_1 \frac{1}{\rho} \frac{\partial p}{\partial x} - \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right] \, dx \, dy \]

an identity is needed for this (see pg. 38-39)

\[ \iint w \frac{\partial \phi}{\partial x} \, dx \, dy = - \iint \frac{\partial \phi}{\partial x} G \, dx \, dy + \oint_{\delta \Omega} n_x w G \, ds \]

\[ \iint w \frac{\partial \phi}{\partial y} \, dx \, dy = - \iint \frac{\partial \phi}{\partial y} G \, dx \, dy + \oint_{\delta \Omega} n_y w G \, ds \]
Now,

$$\int \int w_i \left( \frac{\partial P}{\partial x} \right) dx dy = - \int \int \frac{\partial w_i}{\partial x} P dx dy + \oint n_x w_i P ds$$

Let $K = \frac{\partial u}{\partial x}$, $H = \frac{\partial u}{\partial y}$

$$\int \int w_i (\frac{\partial K}{\partial x} + \frac{\partial H}{\partial y}) dx dy = - \int \int (\frac{\partial w_i}{\partial x} K + \frac{\partial w_i}{\partial y} H) dx dy + \oint (n_x w_i K + n_y w_i H) ds$$

:. 

$$\int \int w_i (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) dx dy = - \int \int \left( \frac{\partial w_i}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial w_i}{\partial y} \frac{\partial u}{\partial y} \right) dx dy + \oint (n_x w_i \frac{\partial u}{\partial x} + n_y w_i \frac{\partial u}{\partial y}) ds$$

Use same procedure for 2nd eqn.

Final form for 1st eqn.

$$\int \int [w_i (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})] = - \frac{1}{\rho} \int \int \left( \frac{\partial w_i}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial w_i}{\partial y} \frac{\partial u}{\partial y} \right) P dx dy + \oint (n_x (P + \frac{\partial u}{\partial x}) + n_y \frac{\partial u}{\partial y}) ds = 0$$
\[ 2.5 \]

\[ -\nabla^2 \psi - 5 = 0 \]

\[ -\nabla^2 \psi + \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^4 \psi}{\partial y \partial x} = 0 \]

\[ \nabla \cdot \nabla \psi = \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \]

1. \[ -\left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) - 5 = 0 \]

2. \[ -\left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) + \frac{\partial^4 \psi}{\partial y \partial x} - \frac{\partial^4 \psi}{\partial x \partial y} = 0 \]

\[ -\int w \left( \frac{\partial^4 \psi}{\partial x^2} + \frac{\partial^4 \psi}{\partial y^2} + 5 \right) dxdy = 0 \]

Let \( \xi = \frac{\partial^4 \psi}{\partial x^2} + \frac{\partial^4 \psi}{\partial y^2} \)

\[ \nabla \cdot \xi = \frac{\partial}{\partial x} \left( \frac{\partial^4 \psi}{\partial x^2} \right) + \frac{\partial}{\partial y} \left( \frac{\partial^4 \psi}{\partial y^2} \right) \]

Divergence theorem

\[ \int_{\Omega} \nabla \cdot \xi dxdy = \oint_{\partial \Omega} \hat{n} \cdot \xi ds \]

along with Gradient theorem (2.2.279, 6)

\[ \int_{\Omega} w \frac{\partial \xi}{\partial x} dxdy = -\int_{\partial \Omega} \frac{\partial w}{\partial x} \xi dxdy + \oint_{\partial \Omega} \hat{n} \cdot w \xi ds \]

\[ \int_{\Omega} w \frac{\partial \xi}{\partial y} dxdy = -\int_{\partial \Omega} \frac{\partial w}{\partial y} \xi dxdy + \oint_{\partial \Omega} \hat{n} \cdot w \xi ds \]

from 2

\[ \int_{\Omega} \frac{\partial^4 \psi}{\partial x^2} dxdy = \int_{\Omega} \frac{\partial^4 \psi}{\partial y^2} dxdy \]

Weak form

\[ -\int w \left( \frac{\partial^4 \psi}{\partial x^2} + \frac{\partial^4 \psi}{\partial y^2} + 5 \right) dxdy = -\int w \left( \frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial \psi}{\partial y} \right) dxdy \]

\[ -\oint_{\partial \Omega} \left( \frac{\partial w}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial \psi}{\partial y} + w \frac{\partial \psi}{\partial y} \right) dxdy \]
\[ 2.5 \] (b) 

\[ \begin{align*} 
\text{2) } & - \int \left[ \nabla \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \nabla \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right] \, dx \, dy = 0 \\
& - \int \left[ \nabla \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \cdot \nabla \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right] \, dx \, dy \\
& - \int_{\Gamma} \left( n_x \frac{\partial u}{\partial x} + n_y \frac{\partial v}{\partial y} \right) \, dS = 0 \quad \text{Weak form} \end{align*} \]