Complex Variables: Application of curvilinear coordinates

Complex variables advantageous for problems with boundaries that are ellipses, hyperbolas, nonconcentric circles, etc.

Recall \( \sigma_{xx} + \sigma_{yy} = \nabla^2 \phi = P \)

\( \nabla^2 P = \nabla^4 \phi = 0 \)

\( \phi = r^2 f + g \)

radius complex variables in polar coordinates

difficult to integrate in new curvilinear coord.
better to get stress and displacement

- Functions of complex variables

\( z = x + iy \)

\( f(z) : \frac{z^2 - x^2 + 2ixy + y^2}{z^2} = \frac{|z|^2}{z} \)

\( f(z) = \alpha + i \beta \)

\( = \frac{x^2 - y^2 + i 2xy}{\alpha} \)

\( = \frac{x - iy}{z^2} \)

\( f(z) = \frac{1}{z} = \frac{x - iy}{x^2 + y^2} \)

Separate \( \sinh(z) \) into \( \text{Re} \) and \( \text{Im} \)

\( \sinh(z) = \frac{1}{2}(e^z - e^{-z}) \)
Check cosine derivation

\[ \frac{d}{d\theta}(\cos \theta) = -\sin \theta \]

\[ \frac{d}{d\theta} (\sin \theta) = \frac{df}{d\theta} \sin \theta \rightarrow \text{analytic function} \]

\[ \frac{df}{dx} = \frac{df}{dy} = \frac{df}{dz} \]

\[ f = \alpha + i\beta \]

\[ \frac{df}{dx} = \frac{\partial \alpha}{\partial x} + i \frac{\partial \beta}{\partial x} \]

\[ \frac{df}{dy} = \frac{1}{i} \frac{\partial f}{\partial y} = \frac{1}{i} \left( \frac{\partial \alpha}{\partial y} + i \frac{\partial \beta}{\partial y} \right) \cdot \frac{i}{i} \]

\[ i = -i \frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial y} \]

\[ \frac{\partial \alpha}{\partial x} = \frac{\partial \beta}{\partial y} \quad \text{and} \quad \frac{\partial \alpha}{\partial y} = -\frac{\partial \beta}{\partial x} \]

\[ \text{Cauchy-Riemann conditions} \]

\[ \frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^2 \beta}{\partial y^2} = 0 \]

\[ \frac{\partial^2 \beta}{\partial x \partial y} - \frac{\partial^2 \beta}{\partial y \partial x} = 0 \quad \text{and} \quad -\frac{\partial \alpha}{\partial y \partial x} + \frac{\partial \alpha}{\partial y \partial x} = 0 \]

\[ \text{Re}(f) \text{ or } \text{Im}(f) \text{ satisfies Laplace's eqn.} \]

\[ \therefore f \text{ is a harmonic eqn.} \]

\[ \alpha \text{ and } \beta \text{ are conjugate harmonic functions} \]
\[ \nabla^2 (xy) = 2z \frac{\partial^2}{\partial x^2} = 2 \frac{\partial^2}{\partial x^2} (xy) = 0 \]

Similarly \[ \nabla^2 (xy) = 2 \frac{\partial^2}{\partial y^2} \]

\[ \nabla^4 (xy) = 0 \]

\[ \nabla^4 (x^2 + y^2) = 0 \]

\[ \nabla^4 (r^2) = 0 \]

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See Timoshenko p. 53

\[ \phi = \sinh (ax) \left[ C_1 \cosh (ay) + C_2 \sinh (ay) \right] + C_3 \cosh (ay) \]

**General Stress functions**

\[ \nabla^2 \phi = 0 \]

\[ P = \nabla^2 \phi = \sigma_{xx} + \sigma_{yy} \]

\[ P \text{ has a conjugate } Q \]

\[ \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x} \]

\[ P + iQ \text{ is analytic} \]

Suppose

\[ \int (P + iQ) \, d\bar{z} = 4\hat{\zeta} (x) \]

Integration also analytic

\[ 4\hat{\zeta} (x) = P + iQ = \frac{1}{4} \int P(x) \, d\bar{z} \]

\[ 4\hat{\zeta}' = \frac{1}{4} \hat{\zeta} (x) \]

\[ \frac{\partial \phi}{\partial x} = 4\hat{\zeta} (x) \frac{\partial}{\partial x} \]

\[ \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} + i \frac{\partial}{\partial y} \]

\[ f(x) = p + iq = 4 \int P(x) \, d\bar{z} \]

\[ P + iQ = \frac{1}{4} \left( \frac{\partial P}{\partial y} + i \frac{\partial P}{\partial y} \right) \]

\[ = -i \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \]

**Aside:**

\[ \frac{\partial f}{\partial x} = P + iQ \]

\[ \frac{\partial f}{\partial y} = 2 \frac{\partial P}{\partial x} + i \frac{\partial Q}{\partial x} \]

**Since**

\[ \frac{\partial f}{\partial x} = \frac{\partial Q}{\partial y} \]

Then

\[ \frac{\partial f}{\partial x} = \frac{\partial Q}{\partial y} \quad \frac{\partial f}{\partial y} = -\frac{\partial Q}{\partial y} \]

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\[ \frac{\partial f}{\partial y} = -\frac{\partial Q}{\partial y} \]
Consider
\[ \phi - xp - yg \]

\[ \nabla^2 (\phi - xp - yg) = P - z \frac{\partial P}{\partial x} - 2 \frac{\partial P}{\partial y} = P \frac{\partial^2 P}{\partial x^2} = P - \frac{\partial^2 P}{\partial x^2} = \nabla^2 P \]

= \nabla^2 P - \frac{\partial^2 P}{\partial x^2} = 0

Recall \[ \nabla^2 (xy) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]

Let \( p_i = \phi - xp - yg \)

\[ \phi = p_i + xp + yg \]

Recall:
Analytic Functions \( \frac{d}{dz} = \frac{d}{dx} + \frac{d}{dy} \) in complex plane

Harmonic functions: \( \nabla^2 F = 0 \)

Given \( f(z) = \alpha + i\beta \)

If \( f(z) \) is analytic, \( \alpha \) and \( \beta \) are harmonic

Cauchy-Riemann conditions
\[ \frac{\partial \beta}{\partial x} = \frac{\partial \alpha}{\partial y} / \frac{\partial \beta}{\partial y} = -\frac{\partial \alpha}{\partial x} \]

Stress function \( \phi \)

\( \nabla^2 \phi = \nabla^2 P = 0 \)

\( \therefore P \) is harmonic

Conjugate harmonic function, \( \psi \)

\( f(z) = P + iQ \)

\( \text{Re}[f(z)] = P = 0x + 0y \)

\( \nabla^2 \phi - P = 0 \)

Put in form \( \nabla^2 (\phi - x) = 0 \)

We did this by integrating
\( f(z) \to q(z) = P + iQ = \frac{1}{2} \int (\alpha + i\beta) \, dz \)
This leads to \( \nabla^2 (\phi - xp - yg) = 0 \) harmonic

now \( \nabla^2 (\phi) = 0 \) where \( \phi = \phi - xp - yg \)

\( \phi = 2xp + p_2 \)

\( \phi = 2yg + p_3 \)

we next introduce \( \chi(z) = p_1 + ig \)

\[ \phi = \text{Re}[\chi(z) + \chi(z)^*] \]

\[ \phi = \text{Re}[\frac{\chi(z)}{z} + \chi(z)^*] \]

\[ \phi = \text{Re}[r^2 (\mu + i\sigma)] \]

Displacement in terms of \( \phi \)

Plane stress soln.

\[ E \frac{\partial u}{\partial x} = \sigma_x - \nu \sigma_y \]

\[ E \frac{\partial v}{\partial y} = \sigma_y - \nu \sigma_x \]

\[ \sigma_x = \phi_{yy} \]

\[ \sigma_y = \phi_{xx} \]

\[ \tau_{xy} = -\phi_{xy} \]

\[ E \frac{\partial u}{\partial x} = \phi_{yy} - \nu \phi_{xx} = \sigma_x \]
\[ E \frac{\partial u}{\partial x} = P - \phi_{xx} - \nu \phi_{xx} \quad \text{and} \quad E \frac{\partial v}{\partial y} = P - (1+\nu) \phi_{yy} \]

\[ = P - (1+\nu) \phi_{xx} \]

Use \( P = 4 \frac{\partial \sigma}{\partial x} = 4 \frac{\partial \tau}{\partial y} \)

\[ E \frac{\partial u}{\partial x} = 4 \frac{\partial \sigma}{\partial x} - (1+\nu) \frac{\partial^2 \phi}{\partial x^2} \rightarrow \frac{E}{1+\nu} \frac{\partial u}{\partial x} = 4 \frac{\partial \sigma}{\partial x} - (1+\nu) \frac{\partial^2 \phi}{\partial x^2} \]

\[ \frac{E}{1+\nu} \frac{\partial u}{\partial x} = 2G \frac{\partial u}{\partial x} = \frac{4}{1+\nu} \frac{\partial \sigma}{\partial x} - \frac{\partial^2 \phi}{\partial x^2} \]

\[ 2G \frac{\partial v}{\partial y} = \frac{4}{1+\nu} \frac{\partial \phi}{\partial y} - \frac{\partial^2 \phi}{\partial y^2} \]

\[ G u = \frac{2}{1+\nu} \sigma - \frac{1}{2} \frac{\partial \phi}{\partial x} + f(y) \]

\[ G v = \frac{2}{1+\nu} \sigma - \frac{1}{2} \frac{\partial \phi}{\partial y} + f(x) \]

\[ G \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{2}{1+\nu} \frac{\partial \sigma}{\partial y} - \frac{1}{2} \frac{\partial^2 \phi}{\partial x \partial y} + \frac{d^2 f}{dy^2} + \frac{2}{1+\nu} \frac{\partial \phi}{\partial x} \]

\[ - \frac{1}{2} \frac{\partial^2 \phi}{\partial x \partial y} + \frac{d^2 f}{dy^2} + \frac{d^2 f}{dx^2} = \gamma_{xy} \]

\[ \gamma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{\partial^2 \phi}{\partial x \partial y} + \frac{2}{1+\nu} \left( \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x} \right) + \frac{d^2 f}{dy^2} + \frac{d^2 f}{dx^2} \]

\[ 0 = \frac{df}{dx} + \frac{df}{dy} \]

\( f(y) \) and \( f(x) \) give rigid body motion. Let them equal zero.

Once \( \phi \) has been found:

1) \( P = \nabla^2 \phi = \sigma_x + \sigma_y \)

2) Find \( \phi \) using \( \frac{\partial P}{\partial x} = \frac{\partial \sigma_x}{\partial x} \quad \text{and} \quad \frac{\partial P}{\partial y} = -\frac{\partial \sigma_y}{\partial y} \)

3) Integrate \( f(x) = P \partial x \) to get \( p + \gamma = \frac{1}{2} \int f(x) \partial x \)

4) Determine \( u, v \) for plane stress
Stress and displacements in terms of complex potentials \( Y(z) \) and \( \chi(z) \)

\[
\phi = \text{Re} \left[ \bar{z} Y(z) + \chi(z) \right]
\]

Complex conjugate \( \rightarrow f(z) = \alpha + i\beta \)

\[
\bar{f}(z) = \alpha - i\beta
\]

Ex.: \( f(z) = e^{i\alpha z} = e^{i\ln(x+iy)} = e^{i\ln e^{-ny}} = (\cos(nx) + i\sin(nx))e^{-ny}
\]

\[
\bar{f}(z) = e^{i\alpha \bar{z}} = e^{-i\ln(x-iy)} = (\cos(nx) - i\sin(nx))e^{-ny}
\]

\[
f(z) + \bar{f}(z) = 2\text{Re}[f(z)]
\]

\[
f(z) - \bar{f}(z) = 2i\text{Im}[f(z)]
\]

This allows

\[
2\phi = \bar{z} Y(z) + \chi(z) + z \bar{Y}(z) + \chi(z)
\]

In the expressions for \( u, v \) we need

\( \phi_x \) and \( \phi_y \)

\[
\begin{align*}
z &= x + iy \\
\frac{\partial}{\partial x} &= 1 \quad \frac{\partial}{\partial y} = i \\
\frac{\partial z}{\partial x} &= 1 \quad \frac{\partial \bar{z}}{\partial y} = i \\
\frac{\partial}{\partial x} &= \frac{\partial}{\partial z} \frac{\partial z}{\partial x} \\
\frac{\partial}{\partial y} &= \frac{\partial}{\partial z} \frac{\partial z}{\partial y}
\end{align*}
\]
\[ 2\phi_x - \overline{\psi}(z) + \overline{\psi}'(z) + \overline{\psi}''(z) \]
\[ i\left[2\phi_y - i\left[\overline{\psi}'(z) - \overline{\psi}(z) + \overline{\psi}'(z) + \overline{\psi}''(z)\right]\right] \]

Add above eqns.

\[ 2(\phi_x + i\phi_y) = 2\left[\psi(z) + z\overline{\psi}'(z) + \overline{\psi}(z)\right] \]

From earlier,

\[ 2G(u + iv) = -\left(\frac{\partial \psi}{\partial x} + i\frac{\partial \psi}{\partial y}\right) + \frac{\psi}{1 + \nu} \]

SINCE \( \psi = \rho + ig \)

\[ 2G(u + iv) = -4(z) - 2\overline{\psi}'(z) - \overline{\psi}''(z) + \frac{\psi}{1 + \nu} \]

\[ = \frac{3 - \nu}{1 + \nu} \psi(z) - 2\overline{\psi}'(z) - \overline{\psi}''(z) \]

When \( \psi, \overline{\psi} \) are known; \( u, v \) can be found

For plane strain

Substitute \( \nu \rightarrow \frac{\nu}{1 - \nu} \)

To obtain stress in terms of \( \psi \) and \( \overline{\psi} \)

Start with

\[ \phi_x + i\phi_y = \psi(z) + z\overline{\psi}'(z) + \overline{\psi}'(z) \]

Take \( \frac{\partial}{\partial x} \):

\[ \phi_{xx} + i\phi_{xy} = \psi'(z) + z\overline{\psi}''(z) + \overline{\psi}'(z) + \overline{\psi}''(z) \]

\[ i\frac{\partial}{\partial y} : \phi_{xx} - \phi_{yy} = -\psi'(z) + z\overline{\psi}''(z) - \overline{\psi}'(z) + \overline{\psi}''(z) \]

Subtract

\[ \phi_{xx} + \phi_{yy} = 2\psi'(z) + 2\overline{\psi}'(z) = 4Re[\psi'(z)] \]

Add

\[ \sigma_{xx} - \sigma_{yy} - 2i\sigma_{xy} = 2\left[\psi''(z) + \overline{\psi}''(z)\right] \]

Or take the conjugate

\[ \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\left[z\overline{\psi}''(z) + \overline{\psi}''(z)\right] \]