Theory of Anisotropic Elasticity

Recall the linear anisotropic constitutive law

\[ \sigma_{ij} = C_{ijkl} e_{kl} \]

\[ C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \]

\[ e_{kl} = \frac{1}{2} (u_{k,l} + u_{l,k}) \]

due to symmetry in elastic modulus, \( C_{ijkl} \)

\[ \sigma_{ij} = C_{ijkl} u_{k,l} \]

Consider 2D deformation where \( u_i \) (i=1,2,3) depends only on \( x_1 \) and \( x_2 \)

Without loss of generality, let

\[ u_i = a_i \cdot f(z) \]

\[ z = x_1 + px_2 \]

\( f(z) \) arbitrary function

\( a_i \) and \( p \) are unknowns dependent on material anisotropy and boundary conditions

Since displacements are defined, compatibility is automatically satisfied - equilibrium must be satisfied also

\[ \sigma_{ij} = 0 \]

\[ C_{ijkl} u_{k,l} = 0 \]

Valid for homogeneous materials
differentiate: \( u_k = \alpha_k f(z) \)

\[
u_{ks} = (\delta_{si} + \rho \delta_{s2}) \alpha_k f'(z)
\]

\( \delta_{si} \) - Kronecker delta

\( f'(z) = \frac{df}{dz} \)

now,

\[
C_{ijs} u_{ks} = 0
\]

\[
C_{ijs} (\delta_{si} + \rho \delta_{s2}) (\delta_{s1} + \rho \delta_{s2}) \alpha_k = 0
\]

\[
(C_{iiks} + \rho C_{iiks}) (\delta_{s1} + \rho \delta_{s2}) \alpha_k = 0
\]

\[
[C_{iik1} + \rho (C_{iik2} + C_{iik1}) + \rho^2 (C_{iik2})] \alpha_k = 0
\]

matrix notation

\[
(\begin{bmatrix} \delta & \rho \end{bmatrix}^T \begin{bmatrix} \rho & \rho^2 \end{bmatrix}) D_{\eta} = 0
\]

one solution to this problem is \( \delta \alpha = 0 \)

This is trivial.

Nontrivial solution requires

\[
\begin{bmatrix} \delta & \rho \end{bmatrix}^T \begin{bmatrix} \rho & \rho^2 \end{bmatrix} = 0
\]

gives six roots for \( \rho \) - eigenvalues

associated eigenvectors are \( \alpha_k \)
determine stress and displacement solutions

\[ \sigma_{ij} = C_{ijkl} \delta_{lkm}^{(k)} \]

\[ \sigma_{ij} = C_{ijkl} (\delta_{S1} + \rho \delta_{S2}) \alpha_k f'(z) \]

\[ \sigma_{ij} = (C_{ijkl} + \rho C_{ikjl}) \alpha_k f'(z) \]

\[ \sigma_{ii} = (C_{ikli} + \rho C_{ikl}) \alpha_i f'(z) = (\tau_{ik} + \rho \tau_{ik}) \alpha_i f'(z) \]

\[ \sigma_{zz} = (C_{izk} + \rho C_{izk}) \alpha_z f'(z) = (\tau_{zk} + \rho \tau_{ik}) \alpha_z f'(z) \]

\[ \lambda \text{ The eigenvalues must be imaginary } \rho_\alpha = \rho_\alpha \text{ and } \rho_\alpha = \rho_\alpha \text{. } \]

\[ i = \sqrt{-1} \]

\[ \lambda \text{ this is required for the strain energy to be positive definite. } \]

since \( \alpha \) is real, we take

\[ \nu = \frac{1}{2} \sum_{\alpha=1}^{3} \left( \alpha \alpha f'(z_\alpha) + \overline{\alpha} \alpha f'(\overline{z}_\alpha) \right) \]

\[ \text{Im } \rho_{\alpha} > 0 \]

\[ \rho_{\alpha 3} = \overline{\rho}_{\alpha} \quad \text{complex conjugate} \]

\[ \alpha_{\alpha 3} = \overline{\alpha}_{\alpha} \]

\[ z_{\alpha} = X_\alpha + \rho_{\alpha} \overline{z}_2 \]
Similarly stress can be defined using the previous equations for \( \sigma_{i1} \) and \( \sigma_{i2} \):

Recall \( t_i = \sigma_i n_i \):

\[
\begin{align*}
t_1 &= \sigma_{i1} n_1 + \sigma_{i2} n_2 + \sigma_{i3} n_3 \\
t_2 &= \sigma_{i2} n_1 + \sigma_{i3} n_2 + \sigma_{i3} n_3
\end{align*}
\]

The vector components of stress can be written as

\[
\begin{align*}
(\mathbf{t}_1)_i &= \sigma_{i1} \\
(\mathbf{t}_2)_i &= \sigma_{i2}
\end{align*}
\]

Such that now,

\[
\begin{align*}
\tilde{t}_1 &= \frac{3}{\alpha_{i1}} \sum \left( \mathbf{a} \cdot \mathbf{b} \right) a_i f_a^l(\bar{z}_a) + \left( \mathbf{a} \cdot \mathbf{b} \right) a_i f_a^l(\bar{z}_a) \\
\tilde{t}_2 &= \frac{3}{\alpha_{i2}} \sum \left( \mathbf{a} \cdot \mathbf{b} \right) a_i f_a^l(\bar{z}_a) + \left( \mathbf{a} \cdot \mathbf{b} \right) a_i f_a^l(\bar{z}_a)
\end{align*}
\]

This can be simplified or rewritten in condensed notation as

\[
\begin{align*}
\sigma_{i1} &= -\mathbf{b}_i f'(\bar{z}) \\
\sigma_{i2} &= \mathbf{b}_i f'(\bar{z})
\end{align*}
\]

Where, \( \mathbf{b} = \left( \mathbf{R}_z + \rho \mathbf{T}_z \right) \mathbf{a} = -\frac{1}{\rho} \left( \mathbf{Q} + \rho \mathbf{R} \right) \mathbf{a} \)

Introducing the stress function, \( \psi_i = \mathbf{b}_i f(\bar{z}) \)

\[
\begin{align*}
\sigma_{i1} &= -\psi_{i,2} \\
\sigma_{i2} &= \psi_{i,1}
\end{align*}
\]
the general displacement and stress potential relations are now,

\[ u = \sum_{\alpha=1}^{3} a_\alpha f_\alpha(\bar{z}_\alpha) + \bar{a}_\alpha f_{\alpha+3}(\bar{z}_\alpha)^3 \]

\[ \phi = \sum_{\alpha=1}^{3} b_\alpha f_\alpha(\bar{z}_\alpha) + \bar{b}_\alpha f_{\alpha+3}(\bar{z}_\alpha)^3 \]

and \( b_{\alpha+3} = \bar{b}_\alpha \)

This is the sextic formalism due to Stroh and \( a_\alpha \) and \( b_\alpha \) are the Stroh eigenvectors. The only stress component missing is \( \sigma_{33} \). It is determined in terms of other stress components for the plane strain condition, \( \varepsilon_{33} = 0 \).

How do you obtain the plane \( \sigma \) case?

Application to boundary value problems

Most applications (excluding bimaterials) \( f_\alpha \) has the same functional form. We can then write,

\[ f_\alpha(\bar{z}_\alpha) = f(\bar{z}_\alpha)g_\alpha \]

\[ f_{\alpha+3}(\bar{z}_\alpha) = f(\bar{z}_\alpha)\bar{g}_\alpha \]

\( g_\alpha \) are arbitrary complex constants that must satisfy boundary conditions.

The equation above can now be written as

\[ u = 2\text{Re} \left\{ A \right\langle f(\bar{z}_\alpha) \rangle g_\alpha \}

\[ \phi = 2\text{Re} \left\{ B \right\langle f(\bar{z}_\alpha) \rangle \bar{g}_\alpha \}

\[ \bar{\phi} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \]

\[ \langle f(\bar{z}_\alpha) \rangle = \text{diag}[f(\bar{z}_1), f(\bar{z}_2), f(\bar{z}_3)] \]
Example

Anti-plane deformation

\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \]
\[ \sigma_{ij} = 0 \]
\[ \varepsilon = \frac{1}{2} (U_{ij} + U_{ji}) \]
\[ \sigma_{ij} = C_{ijkl} U_{kj}, s_j = 0 \] must satisfy this equation subject to some set of boundary conditions

For the anti-plane problem, consider special anisotropic materials which satisfy

\[ C_{ij} = C_{55} = C_{24} = C_{25} = C_{16} = C_{36} = 0 \] (Voigt notation)

Monoclinic materials with symmetry along \( x_3 \) satisfy this fairly general

For anti-plane deformation

\[ U_1 = U_2 = 0 \]
\[ U_3 = U(x_1, x_2) \]

\[ \sigma_{ij} = C_{ijkl} U_{ij} + C_{ijkl} U_{i2} \]
\[ \sigma_{31} = C_{55} U_{11} + C_{45} U_{22} \]
\[ \sigma_{32} = C_{45} U_{11} + C_{44} U_{22} \]
\[ \sigma_{33} = C_{35} U_{11} + C_{34} U_{22} \]

equilibrium requires, \( \sigma_{31} + \sigma_{32} = 0 \)

\[ C_{55} U_{11} + 2C_{45} U_{12} + C_{44} U_{22} = 0 \] homogeneous second order differential equation for \( U \)
The stresses are:

\[ \sigma_{xx} = \frac{1}{A_{12}} \left( -\sigma_{zz} \right) \]

\[ \sigma_{yy} = \frac{1}{A_{12}} \left( -\sigma_{zz} \right) \]

\[ \sigma_{zz} = \frac{1}{A_{12}} \left( -\sigma_{zz} \right) \]

and \( \lambda \) is the shear modulus. (This also holds for plane deformation.)

If the material is isotropic, \( G = G = G \), \( G \) is 0.

\[ \sigma_{xx} = \frac{1}{A_{12}} \left( -\sigma_{zz} \right) \]

\[ \sigma_{yy} = \frac{1}{A_{12}} \left( -\sigma_{zz} \right) \]

\[ \sigma_{zz} = \frac{1}{A_{12}} \left( -\sigma_{zz} \right) \]

Substitute \( \sigma_{zz} \) into 2nd order ODE for properties of \( \lambda \).

\( \lambda = \sqrt{G} \) and \( \rho \) must be determined.

\( \lambda = \sqrt{G} - \sqrt{G} > 0 \) to satisfy positive definite.