Identifying the Time of a Step-Change with c^2 Control Charts

by

Gunabushanam Nedumaran

Oracle Corporation
1133 Esters Road, #602
Irving, TX 75061

Joseph J. Pignatiello, Jr.

Department of Industrial Engineering
FAMU-FSU College of Engineering
Florida State University
Florida A&M University
Tallahassee, FL 32310-6046

James A. Calvin

Department of Statistics
Texas A&M University
College Station, TX 77843-3143

Abstract

If a control chart signals a change in the process parameter, identifying the time of change will substantially help the signal diagnostics procedure since it simplifies the search for special causes. In this paper we propose a maximum likelihood estimator for the time of a step-change in a multivariate process mean when the observations follow a multivariate normal distribution. We describe how this estimator can be used to identify the change point when a multivariate c^2 control chart signals a change in the process mean. We illustrate the use of our proposed estimator with an example. We assess the performance of the estimator through computer simulation experiments. The results show that our proposed estimator performs effectively and equally well for all process dimensions and shift magnitudes considered. Thus, the estimator provides process engineers with an accurate and useful estimate of the actual time of the change in the process mean.

Key Words

Statistical process control; Multivariate process; c^2 control charts; Hotelling's T^2; T^2 control charts; Change point estimation; Maximum likelihood estimation; Monte Carlo simulation; Process improvement; Special cause identification.

Introduction

Statistical process control (SPC) charts are tools that are used to monitor the state of a process by distinguishing between common causes and special causes of variability. When a control chart signals that a special cause is present, process engineers initiate a search for the special cause. Process engineers' expertise and knowledge of the process help them in identifying which combination of the many process variables caused the signal. This identification enables the engineers to improve product quality by preventing or avoiding changes in those variables which cause poor quality, and optimizing those variables which can lead to better quality.

Knowing when a process has changed would simplify the search for the special cause by shrinking the time window within which to look for the special cause. Consequently, the special cause can be identified more efficiently, and corrective measures can be implemented sooner.

When several characteristics of a manufactured component are to be monitored simultaneously, multivariate Shewhart-type c^2 control charts can be used (Montgomery 1991, p. 326). As long as the points plotted on the c^2 control chart fall below the upper control limit (*UCL*) of the chart, the process is assumed to operate under a stable system of common causes, and hence, in a state of control. When one or more points exceed the *UCL*, the process is deemed out of control due to one or more special causes and an investigation is carried out to detect these special causes.

It is well known that c^2 control charts can signal a change in the multivariate process mean a substantial amount of time after the change actually occurred. This is especially true for small shifts in the process mean since the average run length of the control chart can be large. Hence, examining the process for special causes only at the time of the signal may be ineffective. Knowing when the change had actually occurred would substantially help the signal diagnostic procedure.

Samuel, Pignatiello and Calvin (1998) considered the issue of identifying the change point when a signal is issued by an $\overline{X}$ control chart. They investigated the performance of a maximum likelihood estimator (MLE) of the change point, due to Hinkley (1970), when the estimator was used with $\overline{X}$ control charts. Their investigation showed that the estimator performed effectively by identifying the time of the change in the process mean even in the case of small mean shifts.

In this paper, we derive a maximum likelihood estimator for the time of a step-change in the multivariate process mean. We investigate the performance of our change point estimator when it is used with c^2 control charts. We illustrate the use of our change point estimator through an example.

This paper is organized as follows. In the next section, we describe the process model and derive the maximum likelihood estimator of the change point in the multivariate process mean. We illustrate the use of the estimator in the following section with an example. We then study the performance of the estimator using Monte Carlo simulation.

Multivariate Process Model

Suppose that a multivariate process is monitored by means of a c^2 control chart on p important quality characteristics. Let $\boldsymbol{X}_{ij} = (X_{ij1}, X_{ij2}, \ldots, X_{ijp})'$ be a $p \times 1$ vector which represents the p characteristics on the jth observation (j = 1, 2, ..., n) in the ith subgroup of size n. Suppose further that when the process is in control, the $\boldsymbol{X}_{ij}$'s are independent and identically distributed, and follow a p-variate normal distribution with mean vector m_0 and covariance matrix S_0. That is, the $\boldsymbol{X}_{ij}$'s are iid $\boldsymbol{N}_{\boldsymbol{p}}(m_0, S_0)$ when the process is in control. We let n denote the subgroup size and we let $\overline{\boldsymbol{X}}_i$ denote the average vector of the ith subgroup. That is,

$$\overline{\boldsymbol{X}}_i = \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{X}_{ij} \ .$$

When the ith subgroup is observed, the statistic

$$\chi_i^2 = n\left(\overline{\boldsymbol{X}}_i - \mu_0\right)' \Sigma_0^{-1}\left(\overline{\boldsymbol{X}}_i - \mu_0\right)$$

has a chi-square distribution with p degrees of freedom (Montgomery 1991, p. 326). This statistic is plotted on a χ^2 control chart with upper control limit (*UCL*) set at $\chi^2_{p,\alpha}$, where $\chi^2_{p,\alpha}$ is the $(1-\alpha)$ th percentile point the chi-square distribution with *p* degrees of freedom and α is the probability of a false alarm for each subgroup plotted on the chart.

We will assume that, when the multivariate process mean changes, there has been a step-change from its in-control value of $\mu = \mu_0$ to an unknown value $\mu = \mu_1$ where $\mu_1 \neq \mu_0$. If χ^2_T exceeds the *UCL* of the χ^2 control chart, we conclude that the step-change in the process mean occurred after some unknown time τ where $0 \le \tau \le T-1$. Hence, we assume that the subgroup averages $\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2, \ldots, \overline{\boldsymbol{X}}_\tau$ came from the in-control process and the subgroup averages $\overline{\boldsymbol{X}}_{\tau+1}, \overline{\boldsymbol{X}}_{\tau+2}, \ldots, \overline{\boldsymbol{X}}_T$ came from the out-of-control process. We further assume that there is no change in the covariance structure and that the process mean remains at the new level μ_1 until the special cause has been identified and removed.

The maximum likelihood estimator of τ can be shown to be the value of *t* for which the statistic M_t attains its maximum. That is,

$$\hat{\tau} = \underset{t}{arg\,max}\ M_t, \qquad t = 0,\ 1,\ \ldots,\ T-1$$

where

$$M_t = (T-t)\left(\overline{\overline{\boldsymbol{X}}}_{t,T} - \mu_0\right)' \Sigma_0^{-1}\left(\overline{\overline{\boldsymbol{X}}}_{t,T} - \mu_0\right) \qquad (1)$$

and where

$$\overline{\overline{X}}_{t,T} = \frac{1}{T-t}\sum_{i=t+1}^{T}\overline{X}_i \qquad (2)$$

is the average of the $T-t$ most recent subgroup averages. The interested reader can see the Appendix for the derivation of this change point estimator. In the next section we illustrate the use of this estimator with an example.

Illustrative Example

We now consider a hypothetical example involving the machining of steel sleeves in which the inside diameter, the outside diameter and the length are the p = 3 important quality characteristics. A c^2 control chart is being used to monitor these characteristics. Based upon historical data, the process is known to be stable and in control, and the observations are established to be multivariate normal vectors with mean vector m_0 and covariance matrix S_0 where

$$m_0 = \begin{bmatrix} 105.0 \\ 150.0 \\ 120.0 \end{bmatrix} \quad \text{and} \quad S_0 = \begin{bmatrix} 9.0 & 9.6 & 5.4 \\ 9.6 & 16.0 & 4.8 \\ 5.4 & 4.8 & 12.0 \end{bmatrix}.$$

Rational subgroups of size n = 5 are periodically formed from the process and the c^2 statistic for the subgroup is plotted on the chart. The probability of a false alarm is set at a = 0.0027. The *UCL* of the c^2 control chart is then

$$UCL_{T^2} = c^2_{3,0.0027} = 14.157\,.$$

The sample averages of 21 subgroups and the corresponding c^2 statistics are shown in Table 1. The control chart has issued an alarm for the 21st subgroup since $c^2_{21} > UCL$. Thus, T = 21.

Table 1. Subgroup average vectors and the c^2 statistics

Subgroup i	$\overline{\boldsymbol{X}}_i'$	c_i^2
1	$[104.757 \quad 150.151 \quad 119.243]$	0.3500
2	$[105.432 \quad 150.252 \quad 122.584]$	3.1890
3	$[104.449 \quad 151.325 \quad 120.496]$	4.1075
4	$[101.822 \quad 146.074 \quad 118.236]$	5.8615
5	$[106.986 \quad 150.596 \quad 121.009]$	4.3125
6	$[106.887 \quad 153.377 \quad 118.408]$	7.2180
7	$[104.486 \quad 148.822 \quad 119.610]$	0.5105
8	$[104.314 \quad 147.559 \quad 120.316]$	2.9110
9	$[103.760 \quad 149.237 \quad 118.594]$	1.3150
10	$[104.488 \quad 149.475 \quad 119.524]$	0.1620
11	$[104.638 \quad 150.276 \quad 120.708]$	1.0780
12	$[102.711 \quad 147.623 \quad 119.969]$	3.9800
13	$[107.061 \quad 152.098 \quad 122.726]$	3.6290
14	$[103.276 \quad 148.987 \quad 119.682]$	2.6700
15	$[105.761 \quad 151.890 \quad 120.036]$	1.3640
16	$[108.153 \quad 151.391 \quad 120.350]$	10.9335
17	$[104.841 \quad 147.558 \quad 119.485]$	4.8650
18	$[104.956 \quad 147.410 \quad 118.942]$	6.8220
19	$[108.306 \quad 151.819 \quad 119.715]$	12.3970
20	$[106.464 \quad 150.938 \quad 118.532]$	5.0165
21	$[109.940 \quad 153.406 \quad 121.605]$	18.1875

Our proposed estimator can now be applied to estimate the change point. To do so, we need to find the reverse cumulative averages $\overline{\overline{\boldsymbol{X}}}_{t,T}$ for t = 0, 1, 2, ..., $T-1$. In our example, the signal was issued at T = 21. Thus, we need

$$\overline{\overline{\boldsymbol{X}}}_{t,21} = \frac{1}{21-t}\sum_{i=t+1}^{21}\overline{\boldsymbol{X}}_i$$

for t = 0, 1, 2, ..., 20. Working with the most recent subgroups first, the reverse cumulative averages are

$$\overline{\overline{\boldsymbol{X}}}_{20,21} = \frac{1}{21-20}\left(\overline{\boldsymbol{X}}_{21}\right) = \begin{bmatrix}109.940 & 153.406 & 121.605\end{bmatrix}'$$

$$\overline{\overline{\boldsymbol{X}}}_{19,21} = \frac{1}{21-19}\left(\overline{\boldsymbol{X}}_{20} + \overline{\boldsymbol{X}}_{21}\right) = \begin{bmatrix}108.202 & 152.172 & 120.069\end{bmatrix}'$$

$$\overline{\overline{\boldsymbol{X}}}_{18,21} = \frac{1}{21-18}\left(\overline{\boldsymbol{X}}_{19} + \overline{\boldsymbol{X}}_{20} + \overline{\boldsymbol{X}}_{21}\right) = \begin{bmatrix}108.236 & 152.054 & 119.951\end{bmatrix}'$$

and so on. All 21 of these reverse cumulative averages are shown in Table 2.

The M_t statistics are then calculated for t = 0, 1, 2, ..., 20 where

$$M_t = \left(21-t\right)\left(\overline{\overline{\boldsymbol{X}}}_{t,T} - \mu_0\right)' \Sigma_0^{-1}\left(\overline{\overline{\boldsymbol{X}}}_{t,T} - \mu_0\right).$$

Working with the most recent reverse cumulative averages we obtain

$$M_{20} = \left(21-20\right)\left(\overline{\overline{\boldsymbol{X}}}_{20,21} - \mu_0\right)' \Sigma_0^{-1}\left(\overline{\overline{\boldsymbol{X}}}_{20,21} - \mu_0\right) = 3.6375$$

$$M_{19} = \left(21-19\right)\left(\overline{\overline{\boldsymbol{X}}}_{19,21} - \mu_0\right)' \Sigma_0^{-1}\left(\overline{\overline{\boldsymbol{X}}}_{19,21} - \mu_0\right) = 3.8007$$

$$M_{18} = \left(21-18\right)\left(\overline{\overline{\boldsymbol{X}}}_{18,21} - \mu_0\right)' \Sigma_0^{-1}\left(\overline{\overline{\boldsymbol{X}}}_{18,21} - \mu_0\right) = 6.2354$$

and so on. All 21 of these M_t statistics are shown in Table 2.

From Table 2, we see that the largest value of M_t occurs at t = 15. So, the estimate of the time of change is $\hat{t} = 15$. Hence, we estimate that the process mean has changed during the time between the formation of subgroups 15 and 16. That is, we estimate that subgroup 15 was the last subgroup from the in-control process and subgroup 16 was the first subgroup from the out-of-control process.

Table 2. Reverse cumulative average vectors and the M_t statistics

t	$\overline{\overline{\boldsymbol{X}}}'_{t,T}$	M_t
0	$[105.404 \quad 150.012 \quad 119.988]$	1.2742
1	$[105.436 \quad 150.005 \quad 120.026]$	1.3840
2	$[105.436 \quad 149.993 \quad 119.891]$	1.5846
3	$[105.491 \quad 149.919 \quad 119.857]$	2.2324
4	$[105.707 \quad 150.145 \quad 119.887]$	2.6874
5	$[105.627 \quad 150.117 \quad 119.887]$	2.1740
6	$[105.543 \quad 149.899 \quad 119.985]$	2.0538
7	$[105.618 \quad 149.976 \quad 120.012]$	2.0942
8	$[105.719 \quad 150.162 \quad 119.989]$	2.0172
9	$[105.882 \quad 150.239 \quad 120.105]$	2.4716
10	$[106.009 \quad 150.309 \quad 120.158]$	2.7918
11	$[106.146 \quad 150.312 \quad 120.103]$	3.5285
12	$[106.528 \quad 150.611 \quad 120.118]$	4.9370
13	$[106.461 \quad 150.425 \quad 119.792]$	5.1909
14	$[106.916 \quad 150.630 \quad 119.807]$	7.3098
15	$[107.108 \quad 150.420 \quad 119.769]$	**8.7092**
16	$[106.899 \quad 150.226 \quad 119.653]$	6.6730
17	$[107.414 \quad 150.893 \quad 119.694]$	6.4799
18	$[108.236 \quad 152.054 \quad 119.951]$	6.2354
19	$[108.202 \quad 152.172 \quad 120.069]$	3.8007
20	$[109.940 \quad 153.406 \quad 121.605]$	3.6375

Process engineers could use this information in their investigation for the special cause responsible for the shift in the process mean. For example, they could review the process log books and study the information recorded during and around the time when subgroup 15 was machined to identify a special cause. They may find, for example, that the boring tool used for machining the inside diameter had lost its alignment during the time subgroup 15 was machined. Upon identifying this as the special cause, the process engineers could correct the misalignment in the tool. They could initiate a series of experiments to investigate whether the existing boring tool should be replaced with a newer and sturdier one.

If the process engineers had examined their records corresponding to subgroup T = 21 only, i.e., the subgroup for which the c^2 control chart issued the signal, they might have failed to identify the special cause or they might have incorrectly identified a special cause. Alternatively, the process engineers could have started examining their records at the time of signal and searched backwards until a special cause was found. But, using our proposed estimator is a more efficient way to search for the special cause. Process engineers could initiate their search at the time suggested by our proposed estimator and they could expand their search window by examining the records corresponding to subgroups machined before and after the estimated change point.

Performance Evaluation

In this section, we study the performance of our proposed estimator using Monte Carlo simulation. Two performance measures, namely, the average change point estimate and the empirical distribution of the estimated change point around the actual change point are considered.

The observations were assumed to come from a $N_p(m_0,\ S_0)$ distribution when the process is in control. Three process dimensions, namely, p = 2, 5 and 10 were considered. One hundred subgroups of size n = 5 were generated randomly from the in-control distribution. If the c^2 statistic for any of these subgroups exceeded the *UCL,* all data from that subgroup were discarded and replaced with new data. The new c^2 statistic was then re-computed and compared with the *UCL*. This procedure was repeated, as needed, until 100 subgroups from the in-control process had c^2 statistics that did not exceed the *UCL*. Thus, the c^2 control chart did not issue any false alarms.

Starting with subgroup 101, the simulated process mean was changed from m_0 to m_1 by introducing a shift of magnitude l in the in-control mean where

$$l = \sqrt{n\left(m_1 - m_0\right)' \, S_0^{-1}\left(m_1 - m_0\right)} \, .$$

Subgroups were then generated from the out-of-control process until a subgroup's c^2 statistic exceeded its *UCL*, that is, until the control chart issued a genuine alarm signal. The change point estimate was then calculated following that genuine alarm signal using the method described earlier. This procedure was then replicated 10,000 times, and the average of those 10,000 change point estimates, its standard error, and the empirical distribution of the estimated change point around the actual change point were obtained. Five shifts of magnitude l = 1.0, 1.5, 2.0, 2.5 and 3.0 were considered.

The results of this simulation study are presented in the following tables. In Tables 3 - 5 we show *E*(*T*), the time at which the c^2 control chart is expected to issue a genuine signal of a change in the process mean. Since the change had actually occurred following subgroup t = 100, *E*(*T*) = 100 + *ARL*, where *ARL* is the average run length of the control chart for the out-of-control process. The average run length of a control chart is the expected number of subgroups required to detect a change in the process parameter. We also show the average of the change point estimates ($\bar{\hat{t}}$) and its standard error based on 10,000 replications. Since the actual change point was at subgroup t = 100, the average of the change point estimates ($\bar{\hat{t}}$) should be close to 100.

The results in Tables 3 - 5 show that the $\bar{\hat{t}}$ averages are in fact close to the actual change point of t = 100 for all shift magnitudes and for all dimensions considered. Thus, on average, our proposed change point estimator is close to the actual time of change regardless of the values of the shift magnitude and process dimension.

Table 3. Average of the change point estimates for t = 100 and p = 2

	l = 1.0	l = 1.5	l = 2.0	l = 2.5	l = 3.0
$E(T)$	167.34	123.34	109.41	104.51	102.57
$\bar{\hat{t}}$	100.37	100.09	99.87	99.75	99.74
Std. Error	0.0782	0.0384	0.0340	0.0333	0.0275

Table 4. Average of the change point estimates for t = 100 and p = 5

	l = 1.0	l = 1.5	l = 2.0	l = 2.5	l = 3.0
$E(T)$	214.38	144.42	117.94	108.07	104.16
$\bar{\hat{t}}$	100.55	100.07	99.99	99.85	99.75
Std. Error	0.0734	0.0385	0.0253	0.0288	0.0332

Table 5. Average of the change point estimates for t = 100 and p = 10

	l = 1.0	l = 1.5	l = 2.0	l = 2.5	l = 3.0
$E(T)$	259.89	172.32	131.25	114.06	106.91
$\bar{\hat{t}}$	100.65	100.19	100.03	99.91	99.87
Std. Error	0.0780	0.0372	0.0267	0.0261	0.0267

For example, when p = 10 and l = 1.0, the c^2 control chart on average will signal the change in the process mean on subgroup 260 when the actual change had occurred after the 100th subgroup. If the process engineers only search their databases for special causes that may have occurred around the time of the signal, it is very unlikely that they will find the real one. If our proposed estimator were used instead, the process engineers would correctly conclude, on average, that the shift had occurred after the 100th subgroup and consequently have a much better chance of identifying the special cause.

In Tables 6 - 8 we show the empirical distribution of the change point estimates around the actual change point. The estimated probability that the change point estimate is within m subgroups of the actual change point is shown in Tables 6 - 8 for several values of m based on 10,000 replications.

For example, consider the case when p = 2 and l = 1.0. In Table 6 we see that of a total of 10,000 replications carried out, the change point was identified correctly in 25% of the replications. The change point was estimated to be within ± 1 subgroup of the actual change point in 46% of the replications, and to be within ± 4 subgroups of the actual change point in 74% of the replications.

Table 6. Empirical distribution of $\hat{t}$ around t for p = 2

	l = 1.0	l = 1.5	l = 2.0	l = 2.5	l = 3.0
$\hat{P}(\hat{t} = t)$	0.25	0.43	0.61	0.72	0.82
$\hat{P}(\lvert\hat{t} - t\rvert \le 1)$	0.46	0.68	0.83	0.90	0.95
$\hat{P}(\lvert\hat{t} - t\rvert \le 2)$	0.59	0.79	0.92	0.96	0.98
$\hat{P}(\lvert\hat{t} - t\rvert \le 3)$	0.67	0.86	0.95	0.98	0.98
$\hat{P}(\lvert\hat{t} - t\rvert \le 4)$	0.74	0.91	0.97	0.98	0.99
$\hat{P}(\lvert\hat{t} - t\rvert \le 5)$	0.78	0.94	0.98	0.99	
$\hat{P}(\lvert\hat{t} - t\rvert \le 6)$	0.82	0.96	0.99		
$\hat{P}(\lvert\hat{t} - t\rvert \le 7)$	0.85	0.97			
$\hat{P}(\lvert\hat{t} - t\rvert \le 8)$	0.88	0.98			
$\hat{P}(\lvert\hat{t} - t\rvert \le 9)$	0.90				
$\hat{P}(\lvert\hat{t} - t\rvert \le 10)$	0.91				
$\hat{P}(\lvert\hat{t} - t\rvert \le 15)$	0.96				

Table 7. Empirical distribution of $\hat{t}$ around t for p = 5

	l = 1.0	l = 1.5	l = 2.0	l = 2.5	l = 3.0
$\hat{P}(\hat{t} = t)$	0.24	0.42	0.59	0.71	0.81
$\hat{P}(\lvert\hat{t} - t\rvert \le 1)$	0.45	0.67	0.82	0.91	0.95
$\hat{P}(\lvert\hat{t} - t\rvert \le 2)$	0.57	0.79	0.91	0.96	0.98
$\hat{P}(\lvert\hat{t} - t\rvert \le 3)$	0.66	0.86	0.95	0.98	0.99
$\hat{P}(\lvert\hat{t} - t\rvert \le 4)$	0.72	0.90	0.97	0.99	
$\hat{P}(\lvert\hat{t} - t\rvert \le 5)$	0.77	0.93	0.98		
$\hat{P}(\lvert\hat{t} - t\rvert \le 6)$	0.81	0.95	0.99		

$\hat{P}(\lvert\hat{t}-t\rvert \le 7)$	0.84	0.96			
$\hat{P}(\lvert\hat{t}-t\rvert \le 8)$	0.86	0.97			
$\hat{P}(\lvert\hat{t}-t\rvert \le 9)$	0.89	0.98			
$\hat{P}(\lvert\hat{t}-t\rvert \le 10)$	0.90				
$\hat{P}(\lvert\hat{t}-t\rvert \le 15)$	0.96				

Table 8. Empirical distribution of $\hat{t}$ around t for p = 10

	l = 1.0	l = 1.5	l = 2.0	l = 2.5	l = 3.0
$\hat{P}(\hat{t}=t)$	0.24	0.43	0.57	0.69	0.78
$\hat{P}(\lvert\hat{t}-t\rvert \le 1)$	0.45	0.67	0.81	0.90	0.94
$\hat{P}(\lvert\hat{t}-t\rvert \le 2)$	0.57	0.79	0.90	0.96	0.98
$\hat{P}(\lvert\hat{t}-t\rvert \le 3)$	0.66	0.86	0.95	0.98	0.99
$\hat{P}(\lvert\hat{t}-t\rvert \le 4)$	0.72	0.90	0.97	0.99	
$\hat{P}(\lvert\hat{t}-t\rvert \le 5)$	0.77	0.93	0.98		
$\hat{P}(\lvert\hat{t}-t\rvert \le 6)$	0.81	0.95	0.99		
$\hat{P}(\lvert\hat{t}-t\rvert \le 7)$	0.84	0.96			
$\hat{P}(\lvert\hat{t}-t\rvert \le 8)$	0.86	0.97			
$\hat{P}(\lvert\hat{t}-t\rvert \le 9)$	0.89	0.98			
$\hat{P}(\lvert\hat{t}-t\rvert \le 10)$	0.90				
$\hat{P}(\lvert\hat{t}-t\rvert \le 15)$	0.95				

For p = 2 and l = 1.0, the c^2 control chart on average would signal the change in the process mean after subgroup 167 when the actual change had occurred after the 100th subgroup. That is, the control chart on average signals the change 67 subgroups after the actual change. Thus, seeking special causes at the time of signal might be futile. However, if our proposed change point estimator were used, there would be a 25% chance of identifying the actual change point. Further, there would be a 46% chance that the estimated change point is within ± 1 subgroup of the actual change point and a 74%

chance that the estimated change point is within ± 4 subgroups of the actual change point. Similar performance results can be seen in Tables 7 and 8.

Tables 3 - 8 show that our change point estimator performs equally well for all dimensions and for all shift magnitudes considered. Tables 3 - 5 show that the average of the change point estimates are close to the actual change point for all process dimensions and for all shift magnitudes considered. Tables 6 - 8 show that the empirical distribution of the change point estimates around the actual change point depends only on the shift magnitude and that the distribution is about the same for all process dimensions considered.

Further, it can be shown that the results in Tables 3 - 8 do not depend on the direction of the shift. Thus, the average of the change point estimates are close to the actual change point regardless of the process dimension, shift magnitude and the direction of the shift. The empirical distribution of the estimated change point around the actual change point depends only on the shift magnitude. It is independent of the direction of the shift and appears to be independent of the process dimension.

Conclusions

In this paper we have proposed an estimator for identifying the time of a step-change in a multivariate process mean. We described how the estimator can be used in conjunction with c^2 control charts. When a c^2 control chart signals a change in the process mean, process engineers carry out a search for special causes responsible for the change. Confining the search only to the time of the signal is likely to be ineffective since the actual change may have taken place a substantial amount of time before the signal. This is especially true when there is only a small change in the process mean since the average run length can be quite large. Process engineers can improve the chances of identifying the special cause by using our proposed estimator.

We illustrated the use of our proposed estimator with an example involving a c^2 control chart. Our proposed estimator could also be used following a signal on a multivariate CUSUM or EWMA chart as well. We presented the results of some simulation experiments carried out to evaluate the performance of the estimator. The simulation studies showed that, given a change in the process mean, the estimator performed effectively and equally well in detecting the actual change point for all process dimensions and all shift magnitudes considered. The results further showed that the empirical distribution of the estimates around the actual change point depends only on the shift magnitude. The empirical distribution is independent of the direction of the shift and it appears to be independent of the dimension of the process.

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References

Casella, G. and Berger, R. L. (1990). *Statistical Inference*. Wadsworth and Brooks/Cole, Belmont, CA.

Hinkley, D. V. (1970). "Inference about the Change-Point in a Sequence of Random Variables". *Biometrika* 57, pp. 1-17.

Johnson, R. A. and Wichern, D. W. (1992). *Applied Multivariate Statistical Analysis*. Prentice Hall, Englewood Cliffs, NJ.

Montgomery, D. C. (1991). *Introduction to Statistical Quality Control*. John Wiley & Sons, New York, NY.

Samuel, T. R.; Pignatiello, J. J; and Calvin, J. A. (1998). "Identifying the Time of a Step Change with $\overline{X}$ Control Charts," *Quality Engineering*, Vol. 10, No. 3, pp. 521-527.

Appendix

We consider the derivation of the joint maximum likelihood estimators (MLEs) of the change point t and the changed multivariate process mean m_1 in this appendix. Maximum likelihood estimation methods are discussed, for example, in Casella and Berger (1990).

We assume that the process mean has changed at an unknown time t (i.e., immediately after observing subgroup t) and that the change is detected at time *T*. The logarithm of the likelihood function (apart from a constant) is given by

$$ln\, L\left(t\,,\ \mathrm{m}_1\,|\mathrm{C}\right) = -\frac{n}{2}\left[\sum_{i=1}^{t}\left(\overline{\boldsymbol{X}}_i - \mathrm{m}_0\right)'\ \mathrm{S}_0^{-1}\left(\overline{\boldsymbol{X}}_i - \mathrm{m}_0\right) + \sum_{i=t+1}^{T}\left(\overline{\boldsymbol{X}}_i - \mathrm{m}_1\right)'\ \mathrm{S}_0^{-1}\left(\overline{\boldsymbol{X}}_i - \mathrm{m}_1\right)\right]$$

where $\mathrm{C} = \left[\overline{\boldsymbol{X}}_1\,,\ \overline{\boldsymbol{X}}_2\,,\ \ldots\,,\ \overline{\boldsymbol{X}}_T\right]$. Using well known results (see, e.g., Johnson and Wichern, 1992, pp. 144 - 145), we can write

$$\begin{aligned}
&\sum_{i=1}^{t}\left(\overline{\boldsymbol{X}}_i - \mathrm{m}_0\right)'\ \mathrm{S}_0^{-1}\left(\overline{\boldsymbol{X}}_i - \mathrm{m}_0\right)\\
&= tr\left[\mathrm{S}_0^{-1}\left(\sum_{i=1}^{t}\left(\overline{\boldsymbol{X}}_i - \mathrm{m}_0\right)\left(\overline{\boldsymbol{X}}_i - \mathrm{m}_0\right)'\right)\right] \qquad \text{(A1)}\\
&= tr\left[\mathrm{S}_0^{-1}\left(\sum_{i=1}^{T}\left(\overline{\boldsymbol{X}}_i - \mathrm{m}_0\right)\left(\overline{\boldsymbol{X}}_i - \mathrm{m}_0\right)' - \sum_{i=t+1}^{T}\left(\overline{\boldsymbol{X}}_i - \mathrm{m}_0\right)\left(\overline{\boldsymbol{X}}_i - \mathrm{m}_0\right)'\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=t+1}^{T}\left(\overline{\boldsymbol{X}}_i - \mathrm{m}_1\right)'\ \mathrm{S}_0^{-1}\left(\overline{\boldsymbol{X}}_i - \mathrm{m}_1\right)\\
&= tr\left[\mathrm{S}_0^{-1}\left(\sum_{i=t+1}^{T}\left(\overline{\boldsymbol{X}}_i - \overline{\overline{\boldsymbol{X}}}_{t,T}\right)\left(\overline{\boldsymbol{X}}_i - \overline{\overline{\boldsymbol{X}}}_{t,T}\right)' + \left(T - t\right)\left(\overline{\overline{\boldsymbol{X}}}_{t,T} - \mathrm{m}_1\right)\left(\overline{\overline{\boldsymbol{X}}}_{t,T} - \mathrm{m}_1\right)'\right)\right]
\end{aligned}$$

where

$$\overline{\overline{\boldsymbol{X}}}_{t,T} = \sum_{i=t+1}^{T}\overline{\boldsymbol{X}}_i$$

and *tr* denotes the trace operator. If the change point t were known, the MLE of μ_1 would be $\hat{\mu}_1 = \overline{\overline{X}}_{t,T}$. Hence,

$$\sum_{i=t+1}^{T} \left(\overline{\boldsymbol{X}}_i - \hat{\mu}_1\right)' \Sigma_0^{-1} \left(\overline{\boldsymbol{X}}_i - \hat{\mu}_1\right) = tr\left[\Sigma_0^{-1}\left(\sum_{i=t+1}^{T}\left(\overline{\boldsymbol{X}}_i - \overline{\overline{\boldsymbol{X}}}_{t,T}\right)\left(\overline{\boldsymbol{X}}_i - \overline{\overline{\boldsymbol{X}}}_{t,T}\right)'\right)\right]. \qquad \text{(A2)}$$

Now,

$$\begin{aligned}
&\sum_{i=t+1}^{T}\left(\overline{\boldsymbol{X}}_i - \mu_0\right)\left(\overline{\boldsymbol{X}}_i - \mu_0\right)' \\
&= \sum_{i=t+1}^{T}\left(\overline{\boldsymbol{X}}_i - \overline{\overline{\boldsymbol{X}}}_{t,T} + \overline{\overline{\boldsymbol{X}}}_{t,T} + \mu_0\right)\left(\overline{\boldsymbol{X}}_i - \overline{\overline{\boldsymbol{X}}}_{t,T} + \overline{\overline{\boldsymbol{X}}}_{t,T} + \mu_0\right)' \qquad \text{(A3)} \\
&= \sum_{i=t+1}^{T}\left(\overline{\boldsymbol{X}}_i - \overline{\overline{\boldsymbol{X}}}_{t,T}\right)\left(\overline{\boldsymbol{X}}_i - \overline{\overline{\boldsymbol{X}}}_{t,T}\right)' + (T-t)\left(\overline{\overline{\boldsymbol{X}}}_{t,T} - \mu_0\right)\left(\overline{\overline{\boldsymbol{X}}}_{t,T} - \mu_0\right)'.
\end{aligned}$$

Using (A1), (A2) and (A3) we can show that

$$\begin{aligned}
ln\, L\left(t,\ \mu_1 \mid \chi\right) &= -\frac{n}{2}\ tr\left\{\Sigma_0^{-1}\left[\sum_{i=1}^{T}\left(\overline{\boldsymbol{X}}_i - \mu_0\right)\left(\overline{\boldsymbol{X}}_i - \mu_0\right)' - (T-t)\left(\overline{\overline{\boldsymbol{X}}}_{t,T} - \mu_0\right)\left(\overline{\overline{\boldsymbol{X}}}_{t,T} - \mu_0\right)'\right]\right\} \\
&= -\frac{n}{2}\left[\sum_{i=1}^{T}\left(\overline{\boldsymbol{X}}_i - \mu_0\right)' \Sigma_0^{-1}\left(\overline{\boldsymbol{X}}_i - \mu_0\right) - (T-t)\left(\overline{\overline{\boldsymbol{X}}}_{t,T} - \mu_0\right)' \Sigma_0^{-1}\left(\overline{\overline{\boldsymbol{X}}}_{t,T} - \mu_0\right)\right].
\end{aligned}$$

The MLE of the change point, denoted by $\hat{t}$, is the value of t that maximizes the log-likelihood. Hence,

$$\hat{t} = \underset{t}{arg\, max}\ M_t, \qquad t = 0,\ 1,\ \ldots,\ T-1$$

where

$$M_t = (T-t)\left(\overline{\overline{\boldsymbol{X}}}_{t,T} - \mu_0\right)' \Sigma_0^{-1}\left(\overline{\overline{\boldsymbol{X}}}_{t,T} - \mu_0\right).$$

Biographical Footnote

Dr. Gunabushanam Nedumaran is a consultant with Oracle Corporation. His research interests are in the area of applied statistics and statistical process control. He is an ASQ Certified Quality Engineer. His email address is gnedumar@cs.com.

Dr. Joseph J. Pignatiello, Jr. is an Associate Professor in the Department of Industrial Engineering in the FAMU-FSU College of Engineering at Florida State University and Florida A&M University. Dr. Pignatiello's interests are in quality engineering and include statistical process control, process capability, design and analysis of experiments, robust design and engineering statistics. He serves on the editorial boards of *IIE Transactions*, *Journal of Quality Technology*, and *Quality Engineering.* His email address is pigna@eng.fsu.edu.

Dr. James A. Calvin is Professor and Head of the Department of Statistics at Texas A&M University. Dr. Calvin's interests are in the area of statistical applications and mixed linear models. He serves on the editorial boards of *Biometrics*, the *Journal of the American Statistical Association* and *Communications in Statistics*. His email address is calvin@stat.tamu.edu.