Closed-Loop System Identification for Small Samples With Constraints

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Traditional approaches to closed-loop identification of transfer function models require a sufficiently large data set and model forms that are general enough while at the same time requiring application of some form of external excitation (a “dither signal”) to the process. In the limit, as the dither signal dominates the control actions, identification becomes easier, but the operation of the process becomes closer to that of an uncontrolled (i.e., open-loop) process, which may be unacceptable. This article proposes a closed-loop system identification procedure that aims to improve model parameter estimates by incorporating prior knowledge about the process in the form of constraints without using a dither signal. A Monte Carlo simulation study is presented to illustrate the small-sample benefits of adding various forms of constraints. It is shown how constraints based on process knowledge, which is relatively easy to gain from prior experience, result in best identified models among the class of constraints considered. In particular, prior knowledge of the input-output delay of the process is shown to be the most important for identifying a process operating in closed-loop. An example based on a real process illustrates the advantages of the proposed method over the dither signal approach.

KEY WORDS: Box–Jenkins transfer function model; Constrained nonlinear least squares; Feedback control; Prior process knowledge.

1. INTRODUCTION

Consider a process that is adjusted in discrete time with a feedback controller. In many industrial settings, it is of interest that the unknown model of the process be identified in a closed loop, that is, while the process is being controlled. Following the control systems literature, we refer to the identification problem as finding both the true model form and the true parameter values of the model of the process; thus it includes estimation.

Open-loop system identification experiments—that is, observing the process when no feedback control is exercised and when the input (the controllable factor) is persistently exciting the process (Söderström, Ljung, and Gustavsson 1976)—is often the preferred way to identify the model of a process. This is not feasible or cost-effective in industrial processes that are unstable and drift off-target dangerously or that produce expensive scrap when uncontrolled. The main difficulty with closed-loop identification is that the data collected during closed-loop operation are less informative than open-loop data, because the control input level is frequently a linear function of the output (and thus the input is not persistently exciting the process). Our purpose in this article is to identify and estimate the model of a process when a feedback controller is in operation, so that, according to the identified model, the controller can be redesigned in such a way that the quality characteristic under control better achieves a given objective.

A common approach for closed-loop process identification is to introduce an additional random signal in the input—a so-called “dither” signal—in a closed-loop process (Box and MacGregor 1974, 1976). But unfortunately, this can be too expensive if the cost of changing the controllable factor or the cost of running the process off-target is high, common conditions that justify the action of a controller (Box and Luceno 1997). This article presents a new method for identifying transfer function models that works under closed-loop operating conditions and does not require a dither signal.

The identifiability conditions developed in the control theory literature (Ljung, Gustavsson, and Söderström 1974; Söderström, Gustavsson, and Ljung 1975; Söderström et al. 1976) are useful for ensuring that the input is sufficiently exciting the process so that the model parameters can be estimated from closed-loop data. But using the identifiability conditions in practical settings is limited, for two main reasons. First, they are asymptotic conditions, and thus they guarantee convergence to the true model only as the sample size tends to infinity. The main consequence of this is that even when the identifiability conditions are satisfied, the quality of the closed-loop parameter estimates obtained with finite samples is usually poor. It has been shown that a dither signal added to the control input can result in improved parameter estimates from small closed-loop data sets (see, e.g., MacGregor and Fogal 1995). However, the limitations of the open-loop experiments also apply to the use of a dither signal; frequent changes in the input, when excessive, makes the system operate practically in open-loop, which can be undesirable during production.

The second limitation of the identifiability conditions is that the assumed model structure must be sufficiently general to contain the true process model as a particular case. It is worth pointing out the severity of this second condition. If the assumed model does not contain the true model, the true model cannot be identified. More precisely, model misspecification due to not including the true model (underspecification) results in biased or nonunique estimates of the true model. Therefore, in this article we concentrate only on the case where...
the true model is included in the assumed model. Because the true model form is unknown, the only way to guarantee that this condition is satisfied is to use increasingly larger models. But model misspecification due to selecting a larger model than necessary (overspecification) is contrary to the parsimony principle, which favors models with fewer parameters because it causes inflation in the variance of the predictions.

In this study we propose a model identification and estimation approach that uses prior knowledge about the process to improve the parameter estimates obtained from closed-loop data without introducing any external excitation into the system. The proposed approach, which we call the constrained least squares (CLS) approach, incorporates the prior knowledge about the process as constraints on the parameters. We study the bias and variance of the parameter estimates in a simulation example. In all illustrations, we consider only processes for which the identifiability conditions hold.

The rest of the article is organized as follows. Section 2 discusses closed-loop identification and parameter estimation of Box–Jenkins (BJ) transfer function models using the proposed approach. We adopt a BJ model form, because this is the most natural and most common form in the quality control literature. Section 3 reviews possible forms of process information that can be specified as different types of constraints. Section 4 shows how adding constraints improves process identifiability. Sections 5 and 6 present the main results of the article. Section 5 studies the small-sample benefits of adding constraints on the properties of the parameter estimates by Monte Carlo simulation and by considering different model and sample sizes. Section 6 illustrates the application of the approach on real process data and compares it with the identification method based on a dither signal.

2. CLOSED–LOOP IDENTIFICATION OF BOX–JENKINS TRANSFER FUNCTION MODELS

In this article we use BJ transfer function models to identify and estimate controlled processes. The BJ model form is the most natural way to model dynamic–stochastic processes because of its independent characterization of the transfer function and disturbance models giving a signal-plus-noise interpretation.

Denoting the level of the input (the controllable factor) by \( U_t \) and the output (the response variable) deviation from target by \( Y_t \), a BJ transfer function model that relates the input to the output is given as

\[
Y_t = b(B) U_t + \frac{\theta(B)}{\phi(B)} B^d \epsilon_t, \tag{1}
\]

where \( B \) is the backshift operator (i.e., \( BY_t = Y_{t-1} \)), the first term on the right gives the process dynamics, and the second term on the right is the disturbance or noise model that represents the effect of all sources of variation at the output. The parameter \( k \geq 0 \) is the input–output delay, \( d \geq 0 \) is the degree of integration of the disturbance, and \( \{\epsilon_t\} \) is a white noise process this is iid with mean 0 and variance \( \sigma_\epsilon^2 \). We refer to (1) as the assumed process model, denoted as \( M(\beta) \). The model polynomials are defined as

\[
\begin{align*}
b(B) &= b_1 B + b_2 B^2 + \cdots + b_n B^n, \\
a(B) &= 1 - a_1 B - \cdots - a_n B^n, \\
\phi(B) &= 1 - \phi_1 B - \cdots - \phi_n B^n, \\
\theta(B) &= 1 - \theta_1 B - \cdots - \theta_n B^n.
\end{align*}
\]

The degrees or orders of these polynomials \( (n_a, n_b, n_\phi, \text{ and } n_\theta) \) and \( k \) and \( d \) are selected by the user. Therefore, the model parameters can be given in a \( (p \times 1) \) parameter vector as

\[
\beta' = (a_1, \ldots, a_{n_a}, b_1, \ldots, b_{n_b}, \phi_1, \ldots, \phi_{n_\phi}, \theta_1, \ldots, \theta_{n_\theta}).
\]

where \( p = n_a + n_b + n_\phi + n_\theta \).

In model (1), additional disturbance characteristics, such as a process offset or a drift, also can be included,

\[
Y_t = \alpha + \frac{\delta}{(1 - B)^d} + \frac{B^d b(B)}{a(B)} U_t + \frac{\theta(B)}{\phi(B)(1 - B)^d} \epsilon_t, \tag{4}
\]

where \( \delta \) is a drift constant useful in modeling possible trends in the process mean, such as tool wear, which frequently occurs in machining operations. The value of \( \alpha \) represents a process offset that determines the mean deviation from target in case the process is uncontrolled and \( \delta = 0 \).

Closed-loop identification with BJ models has been studied by various authors. Box and MacGregor (1976) applied the results of Söderström et al. (1976) to processes under minimum mean squared error (MMSE) control. They assumed that the true input–output delay and model orders were known, and showed that the precision of the parameter estimates can be improved by the addition of a dither signal. Lučeno (1997) investigated the case of switching between multiple PI controllers during the operation of the process and the effect that this has on the quality of parameter estimates, but also assumed that the correct model orders and the delay were known. More recent approaches to closed-loop identification, the projection method (Forssell and Ljung 2000) and the two-stage method (Van Den Hof and Schrama 1993), break down the problem into two open-loop identification problems.

The idea of applying prior process knowledge in closed-loop identification to improve parameter estimates was discussed by Box and MacGregor (1974) for the cases of prior knowledge of the transfer function dynamics and of the disturbance model. MacGregor and Fogal (1995) investigated the benefits of knowing the disturbance model by applying noise model prefilters in identification. Ljung and Forssell (1998) showed that complete knowledge of the true disturbance model is sufficient for parameter identifiability. Pan and Del Castillo (2001) considered the stationarity and invertibility conditions as a form of prior knowledge and showed by example that closed-loop identification is possible by using these conditions. Although there have been several studies on using prior process knowledge in process identification, to date no systematic study has addressed the relative effectiveness of different forms of prior knowledge.
2.1 Identification and Parameter Estimation of the Process Model

The objective of closed-loop identification is to obtain the representation of a process from a finite set of input–output data, \( Z = \{Y_1, U_1, \ldots, Y_N, U_N\} \), collected during controlled operation of the process. The unknown true process model, designated \( \hat{S} \), can also be represented in the form of a BJ model (4). This model uses the polynomials \( \hat{a}(B), \hat{b}(B), \hat{B}(B), \) and \( \hat{\theta}(B) \); the delay \( \tilde{k} \), and the degree of integration \( \tilde{d} \) for the true process. Throughout the article, we reserve the tilde notation for the true process description.

In the proposed approach, the parameters of the process model are estimated from measured closed-loop output data and the knowledge of the controller by conducting two estimation steps. This approach is sometimes called the “indirect” approach to closed-loop identification. The “direct” approach, in contrast, requires only one estimation step (see, e.g., Ljung 1978).

In the first estimation step, the closed-loop equation of the true process is estimated by fitting an autoregressive moving average (ARMA) time series model to the output data. The closed-loop equation of the true process that obeys the transfer function model (4) is obtained by eliminating the input and representing the output as a time series. To do this, we consider the autoregressive moving average exogenous (ARMAX) representation of this model, given as

\[
\hat{A}(B)Y_t = \tilde{\gamma} + B\tilde{k}\hat{B}(B)U_t + \hat{C}(B)e_t, \tag{5}
\]

where

\[
\hat{A}(B) = (1 - B)^{\tilde{d}}\hat{\phi}(B)\hat{a}(B),
\]

\[
\hat{B}(B) = (1 - B)^{\tilde{d}}\hat{\phi}(B)\tilde{b}(B), \quad \text{and}
\]

\[
\hat{C}(B) = \hat{\theta}(B)\tilde{a}(B)
\]

are the polynomials and \( \tilde{\gamma} = [\hat{a}(1 - B)^{\tilde{d}} + \delta]\hat{\phi}(B)\tilde{\phi}(B) \) is the intercept. Suppose that a linear feedback controller of the form

\[
U_t = \frac{D(B)}{F(B)}Y_t \tag{7}
\]

is acting on the process. By inserting this in (5), we can obtain the ARMA closed-loop equation of the true process as

\[
(\hat{A}(B)F(B) - B\tilde{k}\hat{B}(B)D(B))Y_t = F(B)\tilde{\gamma} + F(B)\hat{C}(B)e_t \quad \text{or} \quad \hat{\Phi}(B)Y_t = \hat{\xi} + \hat{\Theta}(B)e_t, \tag{8}
\]

which can be estimated by fitting an ARMA time series model to the measured output data \( \{Y_t\}_{t=1}^N \). Note that the AR and the MA polynomials of this model are \( \hat{\Phi}(B) = 1 + \hat{\phi}_1B + \cdots + \hat{\phi}_nB^n \) and \( \hat{\Theta}(B) = 1 + \hat{\theta}_1B + \cdots + \hat{\theta}_nB^n \).

In the second estimation step, the parameters of the assumed model (4) are estimated by minimizing the difference between the (estimated) closed-loop equation (8) of the true process and the closed-loop equation of the assumed model. The closed-loop equation of the assumed model is \( (AF - B\xi\tilde{k})Y_t = FY_t + FCE_t \) (here as in later sections, we omit back shift operator for clarity), where the polynomials \( A(B), B(B), \) and \( C(B) \) are obtained by inserting in the expressions (6) the polynomials \( a(B), b(B), \phi(B), \) and \( \theta(B) \) and the degree of integration \( d \) of the assumed model. Therefore, this equation can be written in terms of the BJ model polynomials as

\[
(1 - B)^d\phi(Fa - B\xi\tilde{k})Y_t = F\phi[1 - (1 - B)^d\alpha + \delta] + F\thetaae_t \quad \text{or} \quad \Phi(B)Y_t = \xi + \Theta(B)e_t, \tag{9}
\]

where \( \Phi(B) = 1 + \phi_1B + \cdots + \phi_nB^n \) and \( \Theta(B) = 1 + \theta_1B + \cdots + \theta_nB^n \). Note that for process identifiability, the model must have \( n_\phi \geq n_\theta \) and \( n_\theta > n_\phi \). Thus, the second estimation step involves computing the estimates \( \hat{\Phi} \) by minimizing the sum of \( m = n_\phi + n_\theta \) squared errors obtained by comparing the coefficients of like powers of \( B \) in (9) and in (8),

\[
Q(\beta) = \frac{1}{2} \sum_{i=1}^m c_i^2(\beta)
\quad = \frac{1}{2} \sum_{i=1}^{n_\phi}(\Phi_i - \hat{\Phi}_i)^2 + \frac{1}{2} \sum_{i=1}^{n_\theta}(\Theta_i - \hat{\Theta}_i)^2. \tag{10}
\]

In this study we limit our attention to processes controlled by proportional-integral (PI) controllers. In controller (7), P control is obtained with \( F(B) = 1 \) and \( D(B) = c_1 \), and PI control is obtained with \( F(B) = 1 - B \) and \( D(B) = c_1 + c_2B \), where \( c_1 \) and \( c_2 \) are the controller constants. We assume that under the actions of the feedback controller, the process output is stabilized. Therefore, for the disturbance in (1), we consider models with \( d = 0 \) for a P controlled process and \( d \in \{0, 1\} \) for a PI controlled process (higher values of \( d \) would yield a nonstationary behavior for these controllers).

Considering the possible cases that we can have with PI controllers, the closed-loop model (9) can be further simplified, as shown in Table 1. Note that with \( d = 0 \) and a PI controller, there is a unit root in the MA part of the closed-loop equation. This fact can be exploited to determine the correct value of \( \tilde{d} \) by looking at the MA part \( \Theta(B) \) of the estimated closed-loop equation of the process. Statistical tests that have been developed to detect overdifferencing in autoregressive integrated moving average (ARIMA) model building can be used to determine

<table>
<thead>
<tr>
<th>Degree of integration, ( d )</th>
<th>Controller</th>
<th>AR part, ( \Phi(B) )</th>
<th>Intercept, ( \xi )</th>
<th>MA part, ( \Theta(B) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>P</td>
<td>( \phi(a - B\tilde{k}b_Kp) )</td>
<td>( \phi_0\alpha )</td>
<td>( \theta_0a )</td>
</tr>
<tr>
<td>1</td>
<td>PI</td>
<td>( \phi[1 - B\mua - B\xi\tilde{k}(c_1 + c_2B)] )</td>
<td>( \phi_0\delta )</td>
<td>( \theta_0a )</td>
</tr>
<tr>
<td>0</td>
<td>PI</td>
<td>( \phi[1 - B\mua - B\xi\tilde{k}(c_1 + c_2B)] )</td>
<td>0</td>
<td>( (1 - B)\hat{\theta}_0a )</td>
</tr>
</tbody>
</table>
whether \( \tilde{\Theta}(B) \) contains a unit root (Saikkonen and Luukkonen 1993).

After the value of \( \hat{a} \) is determined, and with the controller known, the appropriate model closed-loop equation in Table 1 is equated to the true process closed-loop equation (8) to compute the sum of squared errors (10), and the minimization of \( Q(\beta) \) yields the parameter estimates \( \hat{\beta} \). The least squares parameter estimation procedure is presented next.

### 2.2 Least Squares Estimation of the Process Model

The parameters in \( \beta \), as defined in (3), are estimated by minimizing the sum of squared errors function (10). After solving for \( \beta \), the remaining parameters \( \delta \) and \( \alpha \) can be estimated from the intercept terms of (9) and (8), because they do not appear in any of the other coefficients.

#### 2.2.1 Unconstrained Nonlinear Least Squares

We refer to estimation without any prior knowledge on the process parameters as the unconstrained case. Note that the error terms \( e(\beta) \) in a BJ model are nonlinear in \( \beta \) due to the presence of the product terms \( \phi_a, \phi_b, \) and \( \theta_a \) in (9). The solution \( \hat{\beta} \) that minimizes (10) can be obtained by solving for the \( (m \times 1) \) vector of errors \( e(\beta) \) using nonlinear least squares techniques. A numerical approach to this is Newton’s method, used to iteratively solve for the linearized form of the error vector (Myers 1990, pp. 426–428). Let \( \beta^{(j)} \) denote the solution obtained in the \( j \)th iteration. In Newton’s method we represent \( e(\beta) \) using a first-order Taylor series around this solution, namely

\[
e(\beta) = e(\beta^{(j)}) + \nabla e(\beta^{(j)})'(\beta - \beta^{(j)}),
\]

where \( \nabla e(\beta) = [\nabla e_1(\beta) \cdots \nabla e_n(\beta)] \) is the Jacobian matrix of \( e(\beta) \). The solution \( \beta^{(j+1)} \) at the next iteration is obtained by setting \( e(\beta) \) in (11) equal to 0 and solving the resulting linear system for \( \beta \). Note that by defining \( y = e(\beta^{(j)}) \), \( X = -\nabla e(\beta^{(j)})y' \), and \( \hat{\beta}^{(j)} = \beta - \beta^{(j)} \), this linear system can be represented as a multiple linear regression model,

\[
y = X\hat{\beta}^{(j)} + e^{(j)},
\]

where \( e^{(j)} \) is the model error. Thus the parameters \( \beta^{(j+1)} \) that minimize the squared norm of \( e^{(j)} \) are obtained from the ordinary least squares solution \( \beta^{(j+1)} = X'X^{-1}X'y = \beta - \beta^{(j)} \), or solving for \( \beta \) and making it equal to \( \beta^{(j+1)} \),

\[
\beta^{(j+1)} = \beta^{(j)} + (X'X)^{-1}X'y.
\]

Starting with an arbitrary vector of estimates, \( \beta^{(0)} \), the iterative procedure is continued using (13) until convergence is achieved. The solution is repeated multiple times, each time with a different starting point randomly selected from the parameter space, in an attempt to obtain the global minimum.

#### 2.2.2 Constrained Nonlinear Least Squares Approach

Prior information on the process parameters is incorporated in the parameter estimation procedure as constraints. Prior information that can be specified as identities for functions of the parameters can be represented as equality constraints on \( \beta \),

\[
v_i(\beta) = 0, \quad i = 1, 2, \ldots, r.
\]

Prior information that can be specified as ranges on functions of the parameters can be represented as inequality constraints on \( \beta \),

\[
g_i(\beta) \leq 0, \quad i = 1, 2, \ldots, s.
\]

Point estimates of the parameters that are obtained from historical data can be incorporated as equality constraints. However, precise estimation of the parameters is not always available in practice. Estimation error can be taken into account through the use of linear inequality constraints that represent confidence intervals on the true values of the parameters.

In the case of equality constraints only, the parameter estimates can be obtained by minimizing a “constrained” sum of squares function,

\[
Q(\beta) = Q(\beta) + \frac{1}{2} \sum_{i=1}^{r} v_i^2(\beta),
\]

using Newton’s method. We note that this solution works regardless of whether the equality constraints (14) are linear or nonlinear functions of the parameters.

In the case of inequality constraints only, the parameter estimates can be obtained by minimizing the squared norm of the model error of the linear system (12) subject to the constraints (15). This is done by defining an objective function \( Q_L(\beta) = \| e^{(j)} \|^2 = \| y - X\hat{\beta}^{(j)} \|^2 \) and solving the minimization

\[
\min Q_L(\beta),
\]

for each iteration \( j \) until convergence is achieved. Because the objective function is quadratic in \( \beta^{(j)} \), this can be solved using quadratic programming techniques when the inequality constraints are linear in \( \hat{\beta}^{(j)} \) (Bazaraa, Sherali, and Shetty 1993). Similar to Newton’s method, this procedure is also repeated from several starting points. In the case with a mix of equality and inequality constraints, we minimize (16) subject to (15). Henceforth, refer to this approach as the CLS approach.

Bayesian methods are also commonly used for incorporating prior information in parameter estimation by modeling the prior information in the form of probability density functions. A Bayesian analysis of the ARMA model (9) requires sampling from nonstandard posterior distributions (see, e.g., Marriott, Ravishanker, Gelfand, and Pai 1996). In addition, to incorporate prior information about functions of parameters the covariance of these parameters must also be modeled properly. In contrast, a CLS approach can incorporate a relatively wide class of prior knowledge while requiring a considerably less computational effort.

### 2.3 Identifiability Conditions

A process \( S \) is said to be parameter identifiable if \( \hat{\beta} \) estimated by minimizing \( Q(\beta) \) is unique and \( M(\hat{\beta}) \) corresponds to \( S \) (Ljung et al. 1974). It has been shown that parameter identifiability in closed loop can be achieved in several ways: (a) by
applying an independent signal in the controller (e.g., a time-varying setpoint or a dither signal) that is persistently exciting of sufficient order (Söderström et al. 1976); (b) by switching between different linear controllers (Ljung et al. 1974); and (c) by having a sufficiently complex controller in the feedback loop (Söderström et al. 1975). In addition, for controller design, Bohlin (1971) has shown that the parameters of a minimum variance controller are identifiable in closed loop provided that only the process input–output delay \( k \) is known.

Söderström et al. (1975) derived the following necessary and sufficient parameter identifiability condition for a true process \( S \) that obeys an ARMAX model form (5) and that is controlled by the controller (7):

\[
\max\{n_F - \tilde{n}_B, n_D + \tilde{k} - \tilde{n}_A\} - \tilde{n}_p \geq 0, \quad (18)
\]

where \( \tilde{n}_p \) is the degree of the polynomial \( \tilde{P} \), which is defined as the common factor between \( \tilde{A}F - B^\tilde{F}BD \) and \( \tilde{C} \). It is evident that this is a condition on both the process and the controller. If (18) is not satisfied, then it is not possible to uniquely identify the true process parameters regardless of the model structure assumed. Condition (18) can be rewritten as

\[
\max\{n_F + \tilde{n}_A, n_D + \tilde{k} + \tilde{n}_B\} - \tilde{n}_p \geq \tilde{n}_A + \tilde{n}_B,
\]

which more clearly states that in the closed-loop model (8), the number of linearly independent equations from the AR part \( \tilde{A}F - B^\tilde{F}BD \), after cancelling the common factors, must be at least equal to the number of unknowns. Note that the MA part \( F\tilde{C} \) does not enter the identifiability condition unless it has a common factor with the AR part.

It has been proved that if the process satisfies the identifiability conditions and the model structure is sufficiently general to contain the true process representation \([i.e., S \in \mathcal{M}(\beta)]\), then the true open-loop representation \( S \) can be obtained in the limit as the sample size \( N \) tends to infinity (Ljung et al. 1974; Ljung 1978).

The parameter identifiability condition for a process that obeys a BJ model form can be derived as follows. From the polynomials (6) written for the true process, we have that \( \tilde{n}_A = \tilde{d} + \tilde{n}_\theta + \tilde{n}_\nu, \tilde{n}_B = \tilde{d} + \tilde{n}_\theta + \tilde{n}_b, \) and \( \tilde{n}_C = \tilde{n}_\theta + \tilde{n}_\nu. \) Furthermore, for the controller (7), we have \( n_D = n_F = 0 \) for a P controller and \( n_D = n_F = 1 \) for a PI controller. Therefore, for the different controller and disturbance cases, we can write the condition (18) for a BJ model form as

\[
\tilde{d} = 0 \quad \text{and P controller:}
\]

\[
\max\{\tilde{n}_A, \tilde{k} + \tilde{n}_B\} - \tilde{n}_p \geq \tilde{n}_A;
\]

\[
\tilde{d} = 1 \quad \text{and PI controller:}
\]

\[
\max\{\tilde{n}_A, I_{PI} + \tilde{k} + \tilde{n}_B - 1\} - \tilde{n}_p \geq \tilde{n}_A;
\]

\[
\tilde{d} = 0 \quad \text{and PI controller:}
\]

\[
\max\{1 + \tilde{n}_A, I_{PI} + \tilde{k} + \tilde{n}_B\} - \tilde{n}_p \geq \tilde{n}_A,
\]

where \( \tilde{n}_A = \tilde{n}_\theta + \tilde{n}_\nu + \tilde{n}_\nu \) and \( I_{PI} = 1 \) with PI control and \( I_{PI} = 0 \) with I control. \( \tilde{P} \) is defined in the same manner as in (18), that is, as the polynomial common to \( \tilde{A}F - B^\tilde{F}BD \) and \( \tilde{C} \).

3. SOME CONSTRAINT TYPES AND THEIR SPECIFICATION

In this section, we review some possible forms of process information that are commonly available in practice and show how they can be represented as equality and inequality constraints as discussed in Section 2.2.

One type of prior knowledge that can be expressed as an equality constraint of the form (14) is the asymptotic gain of the transfer function model (1). The asymptotic gain, denoted by \( \tilde{g} \), is equal to \( Y_\infty \) when \( U_t = 1 \) for all \( t \). For this model, the asymptotic gain can be written using the final value theorem of \( Z \) transforms and the algebraic equivalence between the backshift operator and the \( Z \) transform (i.e., \( B \equiv z^{-1} \)) as

\[
\tilde{g} = Y_\infty = \lim_{t \to \infty} Y_t = \lim_{B \to 1} \frac{\tilde{b}(B)}{\tilde{a}(B)}. \quad (20)
\]

In practice, this limit operation implies that one must wait long enough for the process to settle to observe the steady-state response \( Y_\infty \). This is common practice in analysis methods that study steady-state effects, such as design of experiments (DOE) techniques. Thus the asymptotic gain of a dynamic system can be estimated by fitting a regression model to the steady-state input and output data of the process; the estimated regressor coefficient corresponds to the gain of the process.

As a numerical example, consider the transfer function \( Y_t = \frac{5B}{1 + 5B} U_t \). By letting \( B \to 1 \), the asymptotic gain is calculated as

\[
\tilde{g} = \frac{5(1)}{1 + 5(1)} = 3.33.
\]

Another characteristic of a process that is often known from prior experience with the process is the input–output delay. The process delay \( k \) is defined as the number of whole time periods during which no change in the output is observed in response to a change in the input. The delay can be measured by offline step-response experiments (see, e.g., Björklund and Ljung 2003). The knowledge of the delay can be easily incorporated in identification by simply writing the assumed model with the correct delay.

Knowledge of the disturbance model also can be gained from observing the process in open loop, provided that this is feasible and not too expensive over the period of time where open-loop observations are needed. Usually, 75–100 observations are needed for fitting time series models (see Box, Jenkins, and Reinsel 1994). In a BJ model, the response measured when the input is a constant or 0 is due completely to the disturbance process. For instance, in the case when \( U_t = 0 \), from model (1), we have the response as \( Y_t = \frac{\phi(B)}{\phi(B)(1 - BY \epsilon)}, \) which is identical to the disturbance process. The knowledge on the true values of the coefficients of \( \phi(B) \) and \( \theta(B) \) can be incorporated into the identification procedure by writing an equality constraint for each coefficient in these polynomials.

The usual stationarity and invertibility conditions of the disturbance model [i.e., that the roots of \( \phi(B) \) and \( \theta(B) \) must lie outside the unit circle] and the stability condition of the dynamics transfer function [i.e., that the roots of \( a(B) \) must lie outside the unit circle] can be formulated as inequality constraints. These are standard assumptions of ARMA and transfer function models that usually hold in practice. Assuming an ARMA(2,
2) disturbance model in (1), the stationarity conditions are \(-1 \leq \phi_2 \leq 1, \phi_1 + \phi_2 \leq 1\), and the invertibility conditions are \(-1 \leq \theta_3 \leq 1, \theta_1 + \theta_2 \leq 1\), and \(-\theta_1 + \theta_2 \leq 1\). Assuming a second-order transfer function denominator polynomial, the stability conditions are \(-1 \leq a_2 \leq 1, a_1 + a_2 \leq 1\), and \(-a_1 + a_2 \leq 1\).

4. EFFECTS OF ADDING CONSTRAINTS ON IDENTIFIABILITY

As defined in Section 2.3, a process \(S\) is parameter identifiable if \(\hat{\beta}\) estimated by minimizing \(Q(\beta)\) is unique and \(\mathcal{M}(\hat{\beta})\) corresponds to \(S\) (Ljung et al. 1974). In this section we investigate the effects of introducing equality constraints on the identifiability of the process by looking at the relationship between the convexity properties of the sum of squares function \(Q(\beta)\) and the parameter identifiability.

Consider the true process

\[
(1 + .5B)Y_t = 5BU_t + \epsilon_t, \quad (21)
\]

which is controlled by a P controller \(U_t = K_p Y_t\), where \(K_p = .5\). The true process closed-loop equation (8) is obtained by inserting the controller equation into (21), that is,

\[
(1 + .25B)Y_t = \epsilon_t. \quad (22)
\]

To identify the process, we assume a model of the same form as (21). Therefore, the parameter vector is \(\beta = (a_1, b_1)\), and the delay is \(k = 0\). The assumed model closed-loop equation is obtained by inserting the controller equation into the assumed model, that is,

\[
(1 + \Phi_1 B)Y_t = \epsilon_t, \quad (23)
\]

where \(\Phi_1 = a_1 - K_p b_1\).

It can be seen that (21) is an ARMAX model because it corresponds to (1) with the same denominator polynomials in the transfer function and the disturbance, that is, \(a(\beta) = \phi(\beta)(1 - B)^d = (1 + .5B)\). For a process that obeys an ARMAX model, we have that in the closed-loop equation (9), the \(\phi(\beta)(1 - B)^d\) and \(a(\beta)\) terms of the AR and MA polynomials cancel, and we have

\[
\Phi(\beta) = F(\beta)a(\beta) - B^d b(\beta)D(\beta)
\]

and

\[
\Theta(\beta) = F(\beta)\theta(\beta),
\]

which have all coefficients linear in the parameters. Therefore, the sum of squared errors function (10) is a quadratic (convex) function of \(\beta\).

For this process, \(\tilde{\eta}_A = 1, \tilde{\eta}_B = 1, \tilde{\eta}_C = 0, n_F = 0, n_D = 0, \) and \(\tilde{k} = 0\), and by applying (18) we see that \(\text{max}_{1,1} 1 \geq 2\), and thus the process is not parameter-identifiable. An alternative way to check this is as follows. Because parameter identifiability is equivalent to the existence of a unique minimizing solution to \(Q(\beta)\), a sufficient condition for parameter identifiability of an ARMAX process is that \(Q(\beta)\) is strictly convex. This can be checked by an eigenvalue test for positive definiteness of the Hessian matrix \(H = \left(\frac{\partial^2 Q}{\partial \theta_i \partial \theta_j}\right), i,j = 1, \ldots, p\). For positive definiteness, the eigenvalues \(\lambda_i\) of \(H\) must be \(\lambda_i > 0\) for \(i = 1, \ldots, p\).

The sum of squared errors function computed by comparing (22) and (23) is \(Q(\beta) = \frac{1}{4}c_1^2 = \frac{1}{4}(a_1 - K_p b_1 - .25)^2\), which has the Hessian matrix:

\[
H = \begin{bmatrix}
1 & -K_p \\
-K_p & K_p^2 + 1
\end{bmatrix}.
\]

The eigenvalues of \(H\) are \(\lambda_1 = 0\) and \(\lambda_2 = 1 + K_p\). Thus \(Q(\beta)\) is not strictly convex, which implies that there are infinitely many minimizing solutions and that the process is not parameter-identifiable.

To study the effect of adding a constraint, suppose that we know the gain \(\bar{\gamma} = \frac{5}{1 + 3} = .33\) of the true process. This can be written as a constraint on \(a_1\) and \(b_1\) as \(\frac{b_1}{1 + a_1} = .33\) or in the form of an equality constraint \((1 + 3.33 a_1 - b_1)\). The Hessian matrix of the constrained sum of squared errors function (16) is

\[
H_c = \begin{bmatrix}
1 + 3.33^2 & -K_p - 3.33 \\
-K_p - 3.33 & K_p^2 + 1
\end{bmatrix},
\]

which has the characteristic equation \(\lambda^2 - \lambda(K_p^2 + 13.089) + (1 - 3.33K_p)^2 = 0\). This process is parameter-identifiable as long as the eigenvalues are \(\lambda_1, \lambda_2 > 0\), which is satisfied when the constant term \((1 - 3.33K_p)^2\) is nonzero, that is, when \(K_p \neq \frac{33}{10}\).

\(Q(\beta)\) and \(Q_c(\beta)\) are plotted as functions of \(a_1\) and \(b_1\) in Figure 1 (the logarithmic scale is shown). As can be seen, \(Q(\beta)\) has a locus of minimizing solutions along a straight line, which give equivalent objective function values [Fig. 1(a)]. The line includes the true solution \(a_1 = .5\), \(b_1 = 5\); however, the solution is not unique. This illustrates an instance of a non–parameter identifiable-process. As the result of adding the constraint, the sum of squared errors function becomes strictly convex, resulting in a unique minimizing solution that corresponds to the true process [Fig. 1(b)]. Therefore, the process after adding the constraint is parameter-identifiable.

5. SIMULATION EXAMPLE: SMALL–SAMPLE BENEFITS OF ADDING CONSTRAINTS

In this section we apply the CLS approach on simulated process data to study the small-sample properties of the parameter estimates obtained by using constraints. We consider the process (this is the “true” process that we simulate)

\[
Y_t = \frac{1}{1 - B} + \frac{10B}{1 - .8B}BU_t + \frac{1 - 3B}{1 - B} \epsilon_t, \quad (25)
\]

which is controlled with a PI controller \(U_t = c_1 + c_2 B Y_t\), where \(c_1 = .005\) and \(c_2 = -.01\). The white noise process is \(\epsilon_t \sim \text{iid N}(0, 1)\).

This process obeys the BJ model form (4) with delay \(\tilde{k} = 1\), degree of integration \(d = 1\), drift constant \(\tilde{\delta} = .1\), model polynomial orders \((\tilde{\eta}_d, \tilde{\eta}_b, \tilde{\eta}_c, \tilde{n}_F, \tilde{n}_D) = (1, 1, 0, 0, 0)\), and PI controller polynomial orders \((n_D, n_P) = (1, 1)\). Therefore, in the parameter identifiability condition (19), we have that \(\text{max}_{1,2} \geq 2\); hence the parameter identifiability condition is satisfied by the process. The closed-loop equation (8) of the true process under the given PI controller is

\[
(1 - 1.8B + .75B^2 + .1B^3)Y_t = .02 + (1 - 1.1B + .24B^2)\epsilon_t,
\]

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which is an ARMA(3, 2) model.

The process is identified by assuming the BJ model form (4). We first identify the degree of integration in the disturbance model. Because the closed-loop equation (26) does not have a unit root in the MA polynomial and a PI controller was used, we determine (correctly) the degree of integration as \( d = 1 \) (see Table 1).

The form of the true process model is unknown. However, in selecting the assumed model, we suppose that some prior knowledge on the true model is available to guarantee that the true process is contained in the assumed model. In particular, we consider

\[
\begin{align*}
\text{A small model:} & \\
Y_t = \delta \frac{1}{1-B} + b_1 B + b_2 B^2 & -1 - a_1 B - \theta_1 B \varepsilon_t \\
& + (1 - B)(1 - \phi_1 B) \varepsilon_t \\
& + B^k U_t \\
& + B^k \beta \\
k = 0
\end{align*}
\]

and

\[
\begin{align*}
\text{A large model:} & \\
Y_t = \delta \frac{1}{1-B} + b_1 B + b_2 B^2 + b_3 B^3 + b_4 B^4 & -1 - a_1 B - a_2 B^2 \phi_1 B \varepsilon_t \\
& + (1 - B)(1 - \phi_1 B - \phi_2 B^2) \varepsilon_t \\
& + B^k U_t \\
& + B^k \beta \\
k = 0
\end{align*}
\]

which reflect the cases of a more certain and a less certain prior knowledge of the true model. We can see that both models have larger orders in each polynomial and a smaller delay than the true process (i.e., the true process is contained in both models); thus the identifiability condition is satisfied by both models.

We consider the constraint types that were discussed in Section 3. The symbols given in Table 2 are used to refer to the constraints used in a particular simulation/estimation test. For example, we can incorporate these constraints in the small model as follows. For the delay, we set \( k = 1 \); for the gain, we set \( b_1 + b_2 = \bar{g} \), where \( \bar{g} = 10 \) is the true gain; and for the noise model, we set \( \theta_1 = .3 \) and \( \phi_1 = 0 \). The constraint scenarios considered in the simulations are the single-constraint cases that are shown in Table 2 and the two-constraint cases \((g, n), (k, g), \text{and } (k, n)\).

### 5.1 Estimating the Process Model From the Simulated Realizations

The controlled operation of process (25) was simulated under the given PI controller. To evaluate the effect of using different sample sizes, we generated \( N = 50, 200, \text{and } 400 \) observations in the simulation. We repeated the simulation 1,000 times to obtain a distribution of the parameter estimates.

For each simulated realization, the closed-loop equation (26) was estimated by fitting an ARMA(3, 2) model to the simulated output \( \{Y_t\}_{t=1}^N \), which yielded the estimates of \( \hat{\Phi}(B), \hat{\Theta}(B), \text{and } \hat{\xi} \). (MATLAB’s \textit{armax} function was used for ARMA model fitting; see Ljung 2002.) These estimates were then used in the minimization of the sum of squares function (10) to obtain \( \hat{\beta} \). Note that the closed-loop equation (9) of the assumed models

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconstrained</td>
<td>( u )</td>
</tr>
<tr>
<td>Stationarity, invertibility, stability</td>
<td>( st )</td>
</tr>
<tr>
<td>Known asymptotic gain</td>
<td>( g )</td>
</tr>
<tr>
<td>Known input–output delay</td>
<td>( k )</td>
</tr>
<tr>
<td>Known noise model</td>
<td>( n )</td>
</tr>
</tbody>
</table>
under a PI controller is an ARMA(5, 3) for the small model and an ARMA(8, 5) for the large model; thus $m = 8$ for the small model and $m = 13$ for the large model.

In the scenarios with only equality constraints, the iterative Newton method was applied to solve the nonlinear least squares problem; in the scenarios that include inequality constraints, MATLAB’s `quadprog` function was used to solve the iterative quadratic programming problem. An estimate $\hat{\beta}$ is obtained by solving the problem several times with a different starting point and by retaining the solution giving the smallest objective function value as the global minimum. In each run, the solution is repeated from 100$p$ different starting points.

5.2 Assessing the Benefits of Using Constraints

In this section we discuss the benefit of adding constraints by considering the bias and variance properties of the parameter estimates and the control performance of the parameter estimates.

5.2.1 Bias and Variance of the Parameter Estimates. We define the squared bias of the random vector estimator $\hat{\beta}$ as

$$(\text{Bias } \hat{\beta})^2 = \sum_{i=1}^{p} (E\hat{\beta}_i - \beta_i)^2,$$

that is, as the sum of the squared biases of its individual components. Similarly, we define the variance as the sum of the variances of its individual components,

$$\text{var } \hat{\beta} = \sum_{i=1}^{p} \text{var } \hat{\beta}_i.$$ 

Note that this does not consider the covariance of the estimates. The magnitudes of the covariances were observed to be relatively small, however, and including them in the sum did not change the values significantly. Thus, to have a simpler and easier-to-interpret variance definition, we did not consider the covariances.

The sample squared bias and variance of the simulated realizations of $\hat{\beta}$ obtained with different constraints are plotted in Figure 2. A log scale was used to show the small values more clearly. It can be seen that if a smaller (more parsimonious) model is assumed or a larger sample size is used, then the estimators under all constraints have smaller variance and bias.

The improvement attained in the variance and the bias of the unconstrained estimates by adding constraints can be summarized as follows. Knowledge of the delay is crucial regardless of the model size; it gives the greatest improvement in both the variance and the bias. The knowledge of the gain when the delay is also known (i.e., constraints $k$ and $g$) is better for reducing the variance and the bias when a small model is used; it gives moderate improvement with a large model. In contrast, knowledge of the noise model is better for reducing the variance and the bias with a large model; with a small model, it improves the bias, but inflates the variance. Using the stationarity, invertibility, and stability constraints did not provide significant reduction in the bias or the variance for either model. Among all constraint scenarios, the knowledge of the delay and the gain (i.e., constraints $k$ and $g$) with a small model, and the knowledge of the delay and the noise model (i.e., constraints $k$ and $n$) with a large model give the best identified models that have the smallest variance and bias values.

The bias and variance results also suggest that caution must be exercised when adding constraints so that the system is not

![Figure 2](image-url)

Figure 2. The sample squared bias and variance of the identified models under different constraints and sample sizes. (a) and (c) Identification using a small model. (b) and (d) Identification using a large model.
overconstrained. When we have more certain prior knowledge about the true model and use a small model, the unconstrained case already gives relatively good estimates, and adding constraints can only inflate the variance, as was seen with the noise model constraint; a large model, on the other hand, benefits more from adding constraints and has less risk of being overconstrained.

5.2.2 Control Performance of the Parameter Estimates. The performance of an estimated process model from a control standpoint can be measured by redesigning the controller according to this model in such a way that the process output achieves some optimality criterion. Here we consider minimum mean squared error (MMSE) controllers as the optimal controller, although any other controller objective can be used, as desired. As the performance metric of the designed controller, we consider the variance of the controlled process.

The MMSE control law for a process that obeys the BJ model (1) can be derived by considering the ARMAX form, \( A(B)Y_t = B_1 B'(B)U_t + C(B)e_t \), of this model. Here \( B'(B) \) is defined as \( B'(B) = (1-B)^d \phi(B) \beta(B) \), and the \( A(B) \) and \( C(B) \) polynomials are defined in the same way as in (6). The MMSE control law is given as (Aström 1970)

\[
U_t^* = -\frac{H(B)}{B'(B)L(B)} Y_t. \tag{27}
\]

The \( H(B) \) and \( L(B) \) polynomials are defined as \( H(B) = h_0 + h_1 B + \cdots + h_n B^{n-1} \) and \( L(B) = 1 + l_1 B + \cdots + l_k B^k \), where \( n = \max(n_A, n_B, n_C) \), and their coefficients are obtained by solving the Diophantine identity,

\[
C(B) = A(B)L(B) + B^{k+1}H(B). \tag{28}
\]

For each \( \hat{B} \) and the \( \hat{A}(B), \hat{B}'(B), \) and \( \hat{C}(B) \) polynomials estimated in the simulations, the estimated optimal controller \( \hat{U}_t^* \) can be computed, first by solving (28) for \( \hat{H}(B) \) and \( \hat{L}(B) \) and then by inserting the estimated polynomials in (27). It can be shown that the true optimal controller, or the optimal controller computed from the true process (25), is

\[
U_t^* = \frac{1}{10(1-B)(1+7B)} Y_t.
\]

The variance of the process controlled by the true optimal controller can easily be calculated from the closed-loop equation of the process obtained with this controller; this is equal to

\[
Y_t = (1 + 7B)Y_t
\]

The variance provided by the estimated optimal controller to the variance value 1.49, provided by the true optimal controller. It was assumed that the controllers that provide variances in the range (0, 3) are approximately equal to the true optimal controller, and hence they are “good” controllers. The controllers that provide variances in the range (100, \( \infty \)) were considered “unstable.” Figure 3 gives the percentages of the estimated controllers (out of 1,000) that have good control performance and those that are unstable.

It can be seen that by having more certain prior knowledge of the true model (i.e., using a smaller assumed model) or by using a larger sample size, we can improve the control performance.

![Figure 3](image-url)
The figure also demonstrates the benefits of adding constraints. Knowing the delay, the control performance can be significantly improved with respect to the unconstrained case regardless of the sample size and the model size. Knowing the gain, the improvement in the control performance is larger when a relatively large sample size is used (e.g., when \( N = 200 \) or 400). Knowing the noise model is more advantageous when a relatively small sample size is used (e.g., when \( N = 50 \)).

Among all constraint scenarios, the best control performance is obtained with the knowledge of the delay and the gain (i.e., the constraints \( k, g \)) when the sample size is relatively large and with the knowledge of the delay and the noise model (i.e., the constraints \( k, n \)) when the sample size is relatively small regardless of the model size used.

6. CASE STUDY: GAS FURNACE PROCESS

In this section we compare the performance of the proposed CLS approach and the dither signal approach by applying them on the gas furnace data set given by Box et al. (1994). The data set contains 296 input and output observations collected during open-loop operation of the process. The process input is the gas rate of a gas furnace, and the output is the \%CO\(_2\) concentration at the furnace outlet. The transfer function model estimated from these open-loop data is

\[
Y_t = \frac{-0.4976B - 0.39156B^2 - 0.56424B^3}{1 - 0.51213B} \tilde{B}^2 U_t + \frac{1}{(1 - 0.54505B)(1 - B)} \tilde{e}_t \tag{29}
\]

and \( \tilde{\sigma}_e^2 = 0.6398 \).

To illustrate closed-loop identification, we need a data set for the closed-loop operation of the process. We generate the closed-loop operating data using the reconstructed disturbance data of the process and assuming that a PI controller was in operation. To reconstruct the disturbance data from the open-loop input and output measurements, we assume that the model (29) is the true description of the process; that is, (29) gives the polynomials \( \hat{b}(B), \hat{a}(B), \hat{d}(B), \) and \( \hat{\phi}(B) \); the delay \( \hat{k} \); and the degree of integration \( \hat{d} \) of the process. Denoting the open-loop input and output observations by \( U_{ot} \) and \( Y_{ot} \), the disturbance data can be obtained by using \( N_t = Y_{ot} - \frac{\hat{b}(B)}{\hat{a}(B)} \hat{B}^2 U_{ot} \).

For generating the closed-loop output of the process, we assumed that the transfer function numerator polynomial in (29) is \( \hat{b}(B) = -0.4976B \) (i.e., the higher-order terms were ignored). This simplification is done so that the resulting closed-loop model has a lower order and thus is easier to estimate. From the disturbance data, we generated the controlled output sequence \( \{Y_t\}_{t=1}^{296} \) by using \( Y_t = \frac{\hat{b}(B)}{\hat{a}(B)} B^2 U_t + N_t \), where \( \{U_t\} \) is the input sequence according to the PI control law \( U_t = \frac{c_1 + c_2 B}{1 - B} Y'_t \). \( Y'_t \) is the deviation of the output from target \( T \) (i.e., \( Y'_t = Y_t - T \)), where the target is assumed to be 53 (the average of the \( Y_{ot} \) sequence).

Two PI controllers were used, one with \((c_1, c_2) = (1, -0.7)\) and the other with \((c_1, c_2) = (0.05, -0.3)\). The first one, with its larger proportional and integral constants, varies the input more frequently and hence would result in more precise estimates of the closed-loop model than the second one. Figure 4 shows the closed-loop output and input data generated with the first PI controller. This figure also shows the controlled input and output values with dither signals (i.e., the data corresponding to \( S/N \neq 0 \)). The dither signal approach is explained later.

The parameter identifiability of the process is checked by using the condition in (19) that corresponds to \( \hat{a} = 1 \) and a PI controller, that is, \( \max \{\hat{n}_a, \hat{k} + \hat{n}_b\} - \hat{n}_p \geq \hat{n}_\phi + \hat{n}_a + \hat{n}_b \). It can be seen that for this process, \( \hat{n}_a = 1, \hat{n}_b = 1, \hat{k} = 2, \hat{n}_\phi = 1, \hat{n}_0 = 0, \) and \( \hat{n}_p = 0 \) and that this condition is satisfied.

6.1 Closed-Loop Identification Using the CLS Approach

It can be shown that the closed-loop equation of the true process (29) under a PI controller is an ARMA(5, 1). From the generated output data \( \{Y_t\}_{t=1}^{296} \) shown in Figure 4(a) (with \( S/N = 0 \)), we have (correctly) identified this model. The ARMA(5, 1)
model fitted to this data set is

\[
(1 - 1.9515B + 1.2099B^2 + 0.4557B^3)
\]

\[
-0.8476B^4 + 0.3341B^5 \epsilon_t = (1 - 0.6526B) \epsilon_t,
\]

where all estimates are highly significant. (The numbers in the parentheses are the standard errors of the estimates.) An ARMA(5, 1) model was also fitted to the closed-loop data of the second PI controller; in this model the standard errors of the estimates were relatively larger.

Similar to Section 5, we first identify the degree of integration in the disturbance model. Because the MA part of the closed-loop equation (30) does not contain a unit root and the controller is PI, we identify the degree of integration as \( d = 1 \). For identification and estimation of the remaining parameters of the process model the following BJ model form is assumed

\[
Y_t = \frac{b_1 B + b_2 B^2 + b_3 B^3}{1 - a_1 B} \phi N_t + \frac{1 - \theta_1 B}{(1 - B)(1 - \phi_1 B)} \epsilon_t \quad \text{and (31)}
\]

\( k = 0 \).

The parameters of the model are estimated by minimizing the sum of squared errors function (10) that is evaluated with the closed-loop model parameter estimates given in (30). This corresponds to the unconstrained closed-loop identification.

To illustrate the CLS approach, we assume knowledge of the delay, the gain and the noise model of the process. The true values for these constraints are \( k = 2 \) for the delay, \( g = -0.4976/(1 - 0.51213) = -1.0199 \) for the gain, and \( \theta_1 = 0.54505 \) and \( \theta_2 = 0 \) for the noise model. Similar to Section 5, we use the symbols \( k, g, \) and \( n \) to refer to the constraints. The constraint scenarios \( k, (k, g), (k, n), \) and \( (k, n, g) \) were considered.

### 6.2 Closed-Loop Identification Using the Dither Signal Approach

In closed-loop identification with dither signals, an external dither signal, \( d_t \), is added to the feedback input. Provided that the added dither is uncorrelated with \( \epsilon_t \) and is persistently exciting the system, the transfer function and the disturbance models can be identified from the estimated cross-correlation function between the input and the output sequences. This approach is sometimes referred to as the “direct” method for closed-loop identification (Ljung 1978).

After the addition of the dither signal, the input adjustment of the PI controller is

\[
\nabla U_t = (c_1 + c_2 B) Y_t + \nabla d_t,
\]

where \( d_t \) is a pseudorandom binary sequence (PRBS). The signal-to-noise ratio (S/N) with the dither signal is calculated as \( S/N = \text{var}(d_t)/\text{var}(N) \). We considered the S/N ratios \( S/N = 0.1, 1, \) and 5. For identification under each S/N, we added the dither \( d_t \) to the actions of the PI controllers and generated the corresponding closed-loop input and output sequences \( \{U_t\} \) and \( \{Y_t\} \).

Similar to the CLS approach, we assumed the BJ model (31) to identify and estimate the process model. We estimated the parameters of the model from the generated closed-loop output and input data with the added dither signals using the Proc ARIMA function in SAS. Figure 4 shows the generated closed-loop output and input data using the first PI controller and a dither signal with \( S/N = 1 \).

### 6.3 Comparison of the CLS and the Dither Signal Approaches

Model parameter estimates \( \hat{\beta} \) were computed from the closed-loop output data (with \( S/N = 0 \)) using the CLS method and from the closed-loop input and output data under \( S/N = 0.1, 1, \) and 5 using the dither signal method. We summarized these estimates by considering the corresponding estimated dynamics transfer function \( \frac{b_1 B + b_2 B^2 + b_3 B^3}{1 - a_1 B} \) and the estimated disturbance transfer function \( \frac{1 - \theta_2 B}{1 - \theta_1 B} \). Figure 5 shows the unit step response of the estimated and true transfer functions and illustrates how the model estimates improve as more constraints are added or as the S/N ratio of the dither signal is increased.

The two identification approaches were compared on the basis of the costs that they incur during the identification phase and during the operation phase once the process has been identified. The cost of identification and the cost of operation were measured by the variability of the output around the target (the mean squared deviation from target), \( MSE(Y_t) \), and by the variability of the input \( \text{var}(U_t) \) during the identification experiment and during the operation. During the operation phase, an optimal controller was redesigned based on the identified model and was used to adjust the process.

Similar to Section 5, we considered MMSE controllers as the optimal controller. Using expression (27), we computed an estimated MMSE controller \( \hat{U}_t^* \) for each of the models estimated by the two approaches. As a point of reference, we also computed the true MMSE controller of this process. The true MMSE controller uses the parameter estimates obtained from the open-loop data.

The cost of operation was calculated by simulating the process 1,000 times for 296 periods each under the action of these controllers. Table 3 summarizes the cost of identification and the cost of operation of the controllers designed with the two approaches and those of the true MMSE controller. During identification (performed only once), the cost for the estimated controllers is calculated from the closed-loop data of the PI controllers. With the dither signal approach the input varies more, whereas with the CLS approach, no dither signal is added, and thus the output MSE and the input variance remain at the corresponding PI control operation levels. During operation, \( MSE(Y_t) \) and \( \text{var}(U_t) \) are the averages of the values computed from the simulations, and the numbers in the parentheses are the corresponding standard errors. The open-loop column gives the corresponding costs when applying the true MMSE controller.

The two PI controllers represent rather extreme closed-loop experimental conditions for identification with constraints. The first controller varies the input more drastically (i.e., it incurs in a larger cost of identification) and thus results in better parameter estimates, which in turn provides a lower cost of operation. In contrast, the second controller varies the input less
and results in relatively poor parameter estimates, which in turn provide a higher cost of operation.

However, the CLS approach provided a smaller overall cost than the dither signal approach under both controllers. When the process was identified from the closed-loop data using the second PI controller, the CLS approach (under the constraints $k, n, g$) had only a slightly higher operation cost than the dither signal approach (under $S/N = 1$), while incurring in a considerably smaller cost of identification (in identification, $S/N = 1$ requires 20 times as large input variance as that required by $k, n, g$). When the process was identified from the closed-loop data using the first PI controller, the CLS approach achieved a smaller cost of operation than the dither signal approach (under $S/N = .1$) and incurred in a still smaller identification cost when only the delay ($k$) was assumed known. As more constraints were added, the CLS approach (under the constraints $k, n, g$) achieved a smaller cost of operation than the dither signal approach (under $S/N = 1$ and $S/N = 5$) while incurring a much lower identification cost (in identification, $S/N = 1$ requires almost twice the input variance as that required by $k, n, g$). Note that the cost of operation with the constraints $k, n, g$ is almost equal to that of the true MMSE controller.

The results from using either of the PI controllers illustrate that by using the CLS approach when prior knowledge about the process is available, one can identify models with better quality (i.e., with lower cost of operation) than the dither sig-

![Figure 5. The unit step response of the dynamics and disturbance transfer functions of the true and the estimated models. The estimated models are obtained using the CLS and dither signal approaches. The step input is applied at time 1. (a) and (c) The dynamics transfer function. (b) and (d) The disturbance transfer function. [(a) and (b): true; \(-k, g\); \(-k, n\). (c) and (d): true; \(-S/N = .1\); \(-S/N = 5\).]

Table 3. Cost of identification and cost of operation for closed-loop identification of the gas furnace process using the CLS and dither signal approaches

<table>
<thead>
<tr>
<th>Phase</th>
<th>Open loop (true MMSE)</th>
<th>Closed-loop w/ dither signal</th>
<th>Closed-loop w/ constraints (CLS)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$S/N = .1$</td>
<td>$S/N = 1$</td>
<td>$S/N = 5$</td>
</tr>
<tr>
<td>Identification</td>
<td>$MSE(Y_t)$</td>
<td>.10478</td>
<td>.804</td>
</tr>
<tr>
<td>(N = 296)</td>
<td>var($U_t$)</td>
<td>1.151</td>
<td>1.3706</td>
</tr>
<tr>
<td>Operation</td>
<td>$MSE(Y_t)$</td>
<td>.432</td>
<td>.529</td>
</tr>
<tr>
<td>(N = 296)</td>
<td>var($U_t$)</td>
<td>(SE)</td>
<td>(.002)</td>
</tr>
<tr>
<td></td>
<td>(SE)</td>
<td>(.401)</td>
<td>(.400)</td>
</tr>
</tbody>
</table>

Controller design using closed-loop data under the second PI controller ($c_1 = .05$ and $c_2 = -.03$) and open-loop data

<table>
<thead>
<tr>
<th>Phase</th>
<th>Open loop (true MMSE)</th>
<th>Closed-loop w/ dither signal</th>
<th>Closed-loop w/ constraints (CLS)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$S/N = .1$</td>
<td>$S/N = 1$</td>
<td>$S/N = 5$</td>
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<tr>
<td>Identification</td>
<td>$MSE(Y_t)$</td>
<td>.10478</td>
<td>.705</td>
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<tr>
<td>(N = 296)</td>
<td>var($U_t$)</td>
<td>1.151</td>
<td>.106</td>
</tr>
<tr>
<td>Operation</td>
<td>$MSE(Y_t)$</td>
<td>.432</td>
<td>.467</td>
</tr>
<tr>
<td>(N = 296)</td>
<td>var($U_t$)</td>
<td>(SE)</td>
<td>(.002)</td>
</tr>
<tr>
<td></td>
<td>var($U_t$)</td>
<td>14.820</td>
<td>14.683</td>
</tr>
<tr>
<td></td>
<td>(SE)</td>
<td>(.401)</td>
<td>(.401)</td>
</tr>
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</table>
nal approach by introducing much less extra variability into the process (i.e., by incurring in a lower cost of identification).

7. CONCLUSIONS

This article has presented a closed-loop system identification methodology, CLS, that uses prior process knowledge to obtain improved model parameter estimates. A simulation example and a case study demonstrated the benefits of using the CLS approach and various forms of prior process knowledge with different model and sample sizes over the dither signal approach. All investigated cases assumed a model that contains the true process model. The results of the simulation example indicated that regardless of the model and sample size used, the knowledge of the input–output delay \(k\) is crucial in closed-loop identification. It was further shown that when a small model is used, prior knowledge of the delay and the gain \((k, g)\) give best results. Likewise, when a large model is used, knowing the delay and the noise model \((k, n)\) give the best identified models. This is true with respect to both the bias and variance of the estimates and also from a control performance perspective. Therefore, if the objective in modeling a process is to better control it, then it is suggested that more effort be spent in identifying the delay and the asymptotic gain of the process. The case study based on real process data shows that the proposed CLS method provides better-quality models than the dither signal approach without the need to introduce extraneous variability into the process.

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