Chapter 3

3.1 Kinematics of a Continuum

Kinematics refers to the motion of a particle without regard to what causes the motion. Particles in a continuum refer to the infinitesimal volume of a body.

Particle P at \( t=0 \) moves to \( P' \) at \( t=t \).

The position vector, \( \mathbf{r}(t) = x_1(t) \hat{e}_1 + x_2(t) \hat{e}_2 + x_3(t) \hat{e}_3 \)

Every particle moves according

\[
x_i = x_i(X_1, X_2, X_3, t) \quad i = 1, 2, 3
\]

\( \mathbf{x} = \mathbf{x}(\mathbf{X}, t) \rightarrow \text{Pathline} \)

Position of particle \( X_i = \mathbf{x}(\mathbf{X}, 0) \)

\( X_1, X_2, X_3 \rightarrow \text{Position of particles (material coordinates)} \)

\( x_1, x_2, x_3 \rightarrow \text{Spatial coordinates} \)

In a reference configuration, we may use original particle description leading to a Lagrangian description. If we refer to current coordinates at \( t=t \), then it is called Eulerian description.
3.2 Material and Spatial Description

- Consider the variables

  Scalar $\theta$ Temperature
  Vector $\vec{v}$ Velocity
  Tensor $\overline{T}$ Stress

  Then $\theta = \theta_1(X_1, X_2, X_3, t)$
  $\vec{v}_1 = \vec{v}_1(X_1, X_2, X_3, t)$

  $\overline{T} = \overline{T}_1(x_1, x_2, x_3, t)$ $\leftarrow$ Lagrangian

  If $\theta = \theta_1(x_1, x_2, x_3, t)$
  $\vec{v}_1 = \vec{v}_1(x_1, x_2, x_3, t)$

  $\overline{T} = \overline{T}_1(x_1, x_2, x_3, t)$ $\leftarrow$ Spatial or Eulerian
3.3 Material Derivative

\[ \frac{D}{Dt} (\theta) : \text{Material derivative is defined as the time rate of change of quantity (\theta) for a fixed particle.} \]

If \( \theta = \theta_1(X_1, X_2, X_3, t) \) \( \rightarrow \) Material description

\[ \frac{D\theta}{Dt} = \left( \frac{\partial \theta_1}{\partial t} \right)_{x_i \text{ fixed}} \]

If \( \theta = \theta_2(x_1, x_2, x_3, t) \) \( \rightarrow \) Spatial description

\[ \frac{D\theta}{Dt} = \left( \frac{\partial \theta_2}{\partial t} \right)_{x_i \text{ fixed}} \]

\[ = \frac{\partial \theta_2}{\partial x_1} \frac{\partial x_1}{\partial t} + \frac{\partial \theta_2}{\partial x_2} \frac{\partial x_2}{\partial t} + \frac{\partial \theta_2}{\partial x_3} \frac{\partial x_3}{\partial t} + \left( \frac{\partial \theta_2}{\partial t} \right)_{x_i \text{ fixed}} \]

\[ = \frac{\partial \theta_2}{\partial t} + v_1 \frac{\partial \theta_2}{\partial x_1} + v_2 \frac{\partial \theta_2}{\partial x_2} + v_1 \frac{\partial \theta_2}{\partial x_3} \]

\[ \frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + \bar{v} \cdot \nabla \theta \]

Thus, in spatial description, we have an additional term \( \bar{v} \cdot \nabla \theta \) indicating the effect of velocity at that location \( x_i \).
3.4 Acceleration of a particle

It is the material derivative of velocity

Thus in material description

\[ \dddot{x} = \dddot{x}(X, t) \]

\[ \ddot{v} = \left( \frac{\partial x}{\partial t} \right)_{x_i \text{ fixed}} \equiv \frac{D\dddot{x}}{Dt} \]

\[ \dddot{a} = \left( \frac{\partial \ddot{v}}{\partial t} \right)_{x_i \text{ fixed}} \equiv \frac{D\dddot{v}}{Dt} \]

In terms of the spatial description

\[ \dddot{a} = \frac{\partial \ddot{v}}{\partial t} + (\nabla \ddot{v}) \ddot{v} \]

If \( \ddot{v} = \ddot{v}(x_1, x_2, x_3, t) \)

Then \( a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \)
3.5 Displacement Field

\[ \vec{u} = \vec{r}_t - \vec{r}_0 \]

\[ = \vec{x}(\vec{X}, t) - \vec{X} \]

Example:

\[ x_1 = \frac{X_1}{2}, \quad x_2 = X_2, \quad x_3 = X_3 \]

Find out the displacement field

Solution

\[ u_1 = \frac{X_1}{2} - x_1 = -\frac{X_1}{2} \]

\[ u_2 = u_3 = 0 \]
3.6 Kinematics Equation of Rigid Body Motion

(a) Rigid Body Translation  \( \bar{x} = \bar{X} + c(t) \)
\[ \bar{u} = c(t) \text{  No relative displacement} \]

(b) Rigid Body Rotation  \( \bar{x} - \bar{b} = \tilde{R}(t)(\bar{X} - \bar{b}) \)

where  \( \tilde{R}(t) \) is proper orthogonal (rotation) tensor.
\[ \tilde{R}(0) = I \text{  Rotation about particle } \tilde{b} \]

Rotation about origin  \( \bar{x} = \tilde{R}(t)\bar{X} \)
\[
\begin{align*}
\overline{PQ} &= \bar{X}_p - \bar{X}_Q \\
\bar{x}_p - \bar{b} &= \tilde{R}(t)(\bar{X}_p - \bar{b}) \\
\bar{x}_Q - \bar{b} &= \tilde{R}(t)(\bar{X}_Q - \bar{b}) \\
\bar{x}_p - \bar{x}_Q &= \tilde{R}(t)(\bar{X}_p - \bar{X}_Q) \\
\text{or } \Delta x &= \tilde{R}(t)\Delta \bar{X} \\
\Delta x \cdot \Delta x &= (\tilde{R}(t)\Delta \bar{X}) \cdot \tilde{R}(t)\Delta \bar{X} \\
\Delta x \cdot \Delta x &= \Delta \bar{X} \cdot R^T (R\Delta \bar{X}) \\
\Delta x \cdot \Delta x &= \Delta \bar{X} \cdot (R^T R)\Delta \bar{X} = \Delta \bar{X} \cdot I \Delta \bar{X} = \Delta \bar{X} \cdot \Delta \bar{X}
\end{align*}
\]

Results indicate that there is no change in length but only a Rigid Body Rotation
3.6(c) General Rigid Body Motion

\[ \ddot{x} = \hat{R}(t) \left( \dot{X} - \dot{b} \right)_{\text{Rotation}} + c(t)_{\text{Translation}} \]

\[ \ddot{v} = \dot{\hat{R}}(t)(\dot{X} - \dot{b}) + c(t) \]

\[ (\dot{X} - \dot{b}) = \hat{R}^T (\ddot{x} - \ddot{c}) \]

Substituting \[ \ddot{v} = \dot{\hat{R}}R^T (\ddot{x} - \ddot{c}) + \dot{c}(t) \]

However \[ R \cdot R^T = I \]
\[ \dot{R} \cdot R^T + R \cdot R^T = 0 \]
\[ \dot{R} \cdot R^T = -R \cdot R^T = -(\dot{R} \cdot R^T)^T \]

Thus \[ \dot{R} \cdot R^T \] is antisymmetric with \[ \ddot{w} \] as a dual vector. Using this concept
\[ \ddot{v} = \dot{\ddot{w}} \times (\ddot{x} - \ddot{c}) + \dot{c} \]

Assume that \[ \vec{r} = \ddot{x} - \ddot{c} \] then \[ \ddot{v} = \dot{\ddot{w}} \times (\ddot{x} - \ddot{c}) + \dot{c}. \] In this case, \[ \ddot{v} \] is the velocity of a point located at \[ \vec{r} \] rotating with an angular velocity \[ \ddot{w} \] and a translation velocity \[ \dot{c} \]
3.7 Infinitesimal Deformations

Let $P(X), Q(X + dX)$ deform to $P'(x), Q'(x + dx)$ with

\[
\begin{align*}
\bar{u}_p(X) &= x - X \\
\bar{u}_Q(X + dX) &= (x + dx) - (X + dX)
\end{align*}
\]

Define displacement gradient

\[
\nabla \bar{u} = \frac{d\bar{u}}{dx}
\]

\[
\nabla \bar{u} = \begin{pmatrix}
\frac{\partial u_1}{\partial X_1} & \frac{\partial u_2}{\partial X_1} & \frac{\partial u_3}{\partial X_1} \\
\frac{\partial u_1}{\partial X_2} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_3}{\partial X_2} \\
\frac{\partial u_1}{\partial X_3} & \frac{\partial u_2}{\partial X_3} & \frac{\partial u_3}{\partial X_3}
\end{pmatrix}
\]

Now

\[
d\bar{x} = d\bar{X} + (\nabla \bar{u}) d\bar{X}
\]

Then

\[
d\bar{x} = d\bar{X} + \nabla \bar{u} \cdot d\bar{X}
\]
3.7 Example Problem

In this problem we examine how displacement field can be used to find out the deformed configuration given the original configuration.

In order to solve the problem, we need to locate the deformed state of specific particles and lines using the displacement field. In this case, we use points O, A, C and B; lines OA, AC, CB and BA. Define the deformed points by primes and lines as dashed.

Given \( u_1 = kX_2^2 \)
\( u_2 = u_3 = 0 \)

Sol:
\( x_1 = u + X_1 = X_1 + kX_2^2 \)
\( x_2 = X_2; \quad x_3 = X_3 \)
3.7 Example Problem (continue)

Consider the motion of particle B. Find the deformation gradient.

when \( t = 0 \) point B is at \( (X_1, X_2, X_3) = (0,1,0) \)
when \( t = t \) point B is at \( (x_1, x_2, x_3) = (k,1,0) \)

\[
\nabla \mathbf{u} = \begin{bmatrix} 0 & 2kX_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]
3.7 Infinitesimal Deformations

Now

\[ d\vec{x} = d\vec{X} + (\nabla \vec{u}) d\vec{X} \]

\[ d\vec{x} = (I + \nabla \vec{u}) d\vec{X} \]

we may define \( d\vec{x} = F d\vec{X} \)

\[ d\vec{x} \cdot d\vec{x} = (\tilde{F} \cdot d\vec{X})(\tilde{F} \cdot d\vec{X}) \] \( ds \) length of \( dx \)

\[ (ds)^2 = d\vec{X} \cdot (\tilde{F}^T \tilde{F} d\vec{X}) \] \( dS \) length of \( dX \)

If \( \tilde{F} \) is an orthogonal tensor, then \( \tilde{F}^T \tilde{F} = I \), and \( (ds)^2 = (dS)^2 \)

\[ \tilde{F}^T \tilde{F} = (I + \nabla u)^T (I + \nabla u) \]

\[ = I + \nabla u + ((\nabla u)^T + \nabla u^T) \cdot \nabla u \]

\[ \approx I + \nabla u + (\nabla u)^T \cdot \nabla u \]

\[ \equiv I + 2E \]

\[ \tilde{E} = \frac{1}{2} (\nabla u + \nabla u^T) \]

\[ E_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \]

\( \vec{u} \) is displacement

\( \nabla \vec{u} \) is displacement gradient

\( \tilde{F} \) is deformation gradient

\( E \) is infinitesimal strain tensor

symmetric part of displacement gradient
3.8 Geometrical Meaning of $E_{ij}$

Consider two material elements.

**Diagonal Terms**

**Case (i)** Let

\[
\begin{align*}
&dx^{(1)} = \tilde{F} d\vec{X}^{(1)} \\
&dx^{(2)} = \tilde{F} d\vec{X}^{(2)}
\end{align*}
\]

\[
\begin{align*}
&dx^{(1)} \cdot dx^{(2)} = \tilde{F}^T \tilde{F} d\vec{X}^{(1)} \cdot d\vec{X}^{(2)} \\
&= d\vec{X}^{(1)} \cdot d\vec{X}^{(2)} + 2d\vec{X}^{(1)} \cdot Ed\vec{X}^{(2)}
\end{align*}
\]

\[
(ds)^2 - (dS)^2 = 2(dS)^2 \hat{n} \cdot E\hat{n}
\]

Also for the small deformation,

\[
(ds)^2 - (dS)^2 = (ds + dS)(ds - dS) \approx 2dS(dS - dS)
\]

**Thus**

\[
\frac{ds - dS}{dS} = \hat{n} \cdot E\hat{n} = E_{nn} \quad \text{(not a sum)}
\]

$E_{11}$ Elongation in 1 direction, $E_{22}$ in 2 and $E_{33}$ in 3, hence engineering strain \( \frac{ds - dS}{dS} \) is the diagonal term of E in three directions.
3.8 (b) Continued

Off Diagonal Terms

\[ d\vec{X}^{(1)} = dS_1 \hat{m} \]
\[ d\vec{X}^{(2)} = dS_2 \hat{n} \]

\[ \gamma = \frac{\pi}{2} - \theta \text{ = Shear strain} \]
\[ \cos\left(\frac{\pi}{2} - \gamma\right) = \sin \gamma \quad \text{for small } \gamma \]
\[ \therefore \gamma = 2\hat{m} \cdot \vec{E} \cdot \hat{n} \]

\( \gamma \) is shear strain

Thus \( 2E_{12} \) gives the decrease in angle between lines lying between \( x_1 \) and \( x_2 \) directions.
Example 3.8.2 (page 102)

Given:
Assume $k$ is small i.e order of $10^{-4}$

Find: Particle $P(X = \hat{e}_1 - \hat{e}_2)$, find the unit elongation and change in angle (normal and shear strain) for the two material elements: $d\vec{X}^{(1)} = dX_1\hat{e}_1$ and $d\vec{X}^{(2)} = dX_2\hat{e}_2$

Solution:

$$u_1 = k\left(2X_1 + X_2^2\right)$$
$$u_2 = k\left(X_1^2 - X_2^2\right)$$
$$u_3 = 0$$

$$[\nabla u] = k\begin{bmatrix} 2 & 2X_2 & 0 \\ 2X_1 & -2X_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\nabla u] = k\begin{bmatrix} 2 & -2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E = \frac{1}{2}[\nabla u + \nabla u^T] = \begin{bmatrix} 2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$E_{11} = E_{22} = 2k$

$\Rightarrow dX_1$ and $dX_2$ stretch by $2k$ times

$$\frac{ds_1 - ds_1}{ds_1} = E_{11} = 2k \quad \frac{ds_2 - ds_2}{ds_2} = E_{22} = 2k$$

$E_{12} = 0 \Rightarrow$ no shear strain

Since $\vec{x}_i = \vec{X}_i + \vec{u}_i$

$$d\vec{x}^{(1)} = d\vec{X}^{(1)} + [\nabla \vec{u}]d\vec{x}^{(1)}$$

$$= \begin{bmatrix} dX_1 \\ 0 \end{bmatrix} + k\begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dX_1 \\ 0 \end{bmatrix} = dX_1\begin{bmatrix} 1 + 2k \\ 2k \end{bmatrix}$$

$$d\vec{x}^{(2)} = d\vec{X}^{(2)} + [\nabla \vec{u}]d\vec{x}^{(2)}$$

$$= \begin{bmatrix} dX_2 \\ 0 \end{bmatrix} + k\begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dX_2 \\ 0 \end{bmatrix} = dX_2\begin{bmatrix} 1 + 2k \\ -2k \end{bmatrix}$$
3.9 Principal Strain

Since $\tilde{E}$ is real and symmetric, there are three mutually perpendicular $\hat{n}_1, \hat{n}_2$ and $\hat{n}_3$ directions for which is diagonal

$$[E]_{\hat{n}_i} = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix}$$

$\hat{n}_1, \hat{n}_2$ and $\hat{n}_3$ are the principal directions of strain

$E_1, E_2, E_3$ are the three principal strains (maximum and minimum strains)

The characteristic equation for $\tilde{E}$ is

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0$$

and

$$I_1 = E_{11} + E_{22} + E_{33}$$

$$I_1 = \begin{vmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{vmatrix} + \begin{vmatrix} E_{11} & E_{13} \\ E_{13} & E_{33} \end{vmatrix} + \begin{vmatrix} E_{22} & E_{23} \\ E_{23} & E_{33} \end{vmatrix}$$

$I_1, I_2, I_3$ are principal scalar invariants of the strain tensor

$I_1 = \varepsilon = \text{change in unit volume} = \Delta V/V$

$$I_1 = E_{ii} = E_{11} + E_{22} + E_{33} = \frac{\partial u_i}{\partial X_j} = \text{div} \vec{u}$$
3.10 Dilatation

When a body deforms from a volume $dV$ to $dv$ where $V$, $v$ are volumes in $\{X\}$ and $\{x\}$,

Then

$$dv = ds_1 \cdot ds_2 \cdot ds_3$$
$$= dS_1 (1 + E_1) dS_2 (1 + E_2) dS_3 (1 + E_3)$$
$$= dS_1 dS_2 dS_3 (1 + E_1 + E_2 + E_3 + H.O.T)$$
$$= dV (1 + E_1 + E_2 + E_3)$$
Change in Volume $= \Delta(dV) = dv - dV$

$= E_1 + E_2 + E_3 = E_{11} + E_{22} + E_{33}$

$= E_{ii} =$ first scalar invariant of $\tilde{E}$
3.11 Rotation Tensor

\[ d\vec{x} = d\vec{X} + (\nabla \tilde{u})d\vec{X} \]
\[ \frac{1}{2} \left[ (\nabla \tilde{u}) + \nabla \tilde{u}^T \right] = E \]
\[ d\vec{x} = d\vec{X} + (\tilde{E} + \tilde{\Omega})d\vec{X} \]
\[ \frac{1}{2} \left[ (\nabla \tilde{u}) - \nabla \tilde{u}^T \right] = \Omega \]

\( d\vec{X} \) is stretched and rotated to result in \( d\vec{x} \). Stretching comes purely from \( \tilde{E} \), whereas the rotation comes both from \( \tilde{\Omega} \) and \( \tilde{E} \). However, if \( d\vec{X} \) is along the eigenvector direction of \( \tilde{E} \), then change (rotation) in \( d\vec{X} \) comes purely from \( \tilde{\Omega} \). Thus \( \tilde{\Omega} \) denotes the rotation of eigenvector of \( \tilde{\Omega} \). If \( \vec{t}^A \) is dual vector of \( \tilde{\Omega} \), then

\[ \vec{t}^A \times d\vec{X} = \tilde{\Omega} d\vec{X} \]

and

\[ \vec{t}^A = \tilde{\Omega}_{32} \hat{e}_1 + \tilde{\Omega}_{13} \hat{e}_2 + \tilde{\Omega}_{21} \hat{e}_3 \]

where \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \) lie along the principal direction of \( \tilde{E} \).
3.12 Time Rate of Change of a Material Element

We need to compute the rate of change $\frac{d\mathbf{x}}{dt}$ at time $t$. Let us use spatial coordinates $(x_1, x_2, x_3)$ and denote $\frac{D}{Dt} \frac{d\mathbf{x}}{dt} = (\nabla \mathbf{v}) \frac{d\mathbf{x}}{dt}$

Note that in this definition $\nabla \mathbf{v}$ is defined with respect to the spatial coordinate system, e.g. $\frac{\partial \mathbf{v}}{\partial x}$ and not $\frac{\partial \mathbf{v}}{\partial X}$

Velocity gradient $= \nabla \mathbf{v} = v_{i,j}$, where $(), j$ with respect to spatial coordinates. Thus we have

$\begin{bmatrix}
\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\
\frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\
\frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3}
\end{bmatrix}$
3.13 Rate of Deformation Tensor

Let \( \nabla \tilde{v} = \tilde{D} + \tilde{W} \)

Where \( D = \) symmetric part of velocity gradient.
\[
D = \frac{1}{2} \left[ (\nabla \tilde{v}) + (\nabla \tilde{v})^T \right] = \text{Rate of deformation tensor}
\]
\( W = \) anti-Symmetric part of velocity gradient
\[
w = \frac{1}{2} \left[ (\nabla \tilde{v}) - (\nabla \tilde{v})^T \right] = \text{Spin tensor}
\]
Thus
\[
D_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i})
\]
\[
w_{ij} = \frac{1}{2} (v_{i,j} - v_{j,i})
\]

\( \tilde{D} \) describes the rate of change of length of \( d\tilde{x} \)
\( \tilde{W} \) describes the rate of rotation of \( d\tilde{x} \)
3.14 Spin Tensor and Angular Velocity Vector

Since $\tilde{W}$ is anti symmetric, for any vector $\vec{a}$, there is

$$\tilde{W}\vec{a} = \vec{o} \times \vec{a}$$

where

$$\vec{o} = -(W_{23}\hat{e}_1 + W_{31}\hat{e}_2 + W_{12}\hat{e}_3)$$

Consider a vector $d\vec{x}$, then

$$\tilde{W}d\vec{x} = \vec{o} \times d\vec{x}$$

$$\frac{D}{Dt}(d\vec{x}) = (\nabla \vec{v})d\vec{x} = (\tilde{D} + \tilde{W})d\vec{x}$$

$$= \tilde{D}dx + \tilde{W}dx$$

$$= \tilde{D}dx + [\vec{w} \times d\vec{x}]$$

(simply rotate, no length change)

$$2\tilde{W} \Rightarrow \text{vorticity tensor}$$
Problem 3.39

3.39 (a) Find the rate of deformation and spin tensors for the velocity field \( \nu = (\cos t)(\sin \pi x_1)e_2 \)

(b) For the velocity field of part (a), find the rates of extension of the elements 
\[ dx^{(1)} = (ds_1)e_1, \quad dx^{(2)} = (ds_2)e_2, \quad dx^{(3)} = ds_3 / \sqrt{2}(e_1 + e_2) \]

at the origin at \( t=0 \)
Problem 39

3.39 (a) Find the rate of deformation and spin tensors for the velocity field \( v = (\cos t)(\sin \pi x_1)e_2 \)

(b) For the velocity field of part (a), find the rates of extension of the elements
\[ dx^{(1)} = (ds_1)e_1, \quad dx^{(2)} = (ds_2)e_2, \quad dx^{(3)} = ds_3 / \sqrt{2}(e_1 + e_2) \]

at the origin at \( t=0 \)

Ans: With \( v_1 = 0, \quad v_2 = (\cos t)(\sin \pi x_1), \quad v_3 = 0 \), we have

\[
(\nabla v) = \begin{bmatrix} 0 & 0 & 0 \\ \pi \cos t \cos \pi x_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [D] = \begin{bmatrix} 0 & \frac{\pi \cos t \cos \pi x_1}{2} & 0 \\ \frac{\pi \cos t \cos \pi x_1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

At \( t = 0 \) and at \((0,0,0)\)

\[
[D] = \begin{bmatrix} 0 & \frac{\pi}{2} & 0 \\ \frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

for \( dx^{(1)} = ds_1 e_1 \), its rate of extension is equal to \( D_{11} = 0 \),

for \( dx^{(2)} = ds_2 e_2 \) its rate of extension is equal to \( D_{22} = 0 \),

and for \( dx^{(3)} = \frac{ds_3}{\sqrt{2}} (e_1 + e_2) \)

\[
\frac{1}{ds} \frac{D}{Dt} (dt) = \hat{n} \cdot D\hat{n} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{\pi}{2} & 0 \\ \frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \frac{\pi}{2}
\]
3.15 Conservation of Mass

Since $\rho dV$ is the mass of an element with an infinitesimal volume $dV$,

$$\frac{D}{Dt}(\rho dV) = 0 \Rightarrow \rho \frac{D}{Dt}(dV) + dV \frac{D\rho}{Dt} = 0$$

now, dividing by $dV$ and noting that

$$\frac{1}{dV} \frac{D(dV)}{Dt} = D_{11} + D_{22} + D_{33} = \frac{\partial v_i}{\partial t} = \text{div.}\vec{V}$$

$$\Rightarrow \underbrace{\rho \frac{\partial v_i}{\partial t} + D\rho}_{\text{Mass Flux}} = 0$$

Thus Conservation of Mass/Continuity Equation

$$\rho \cdot \text{div}(\vec{v}) + \frac{D\rho}{Dt} = 0$$

In spatial description,

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho$$

Combining the two,

$$\rho \frac{\partial v_i}{\partial x_i} + \rho \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho = 0$$

Thus

$$\rho \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) + \frac{\partial \rho}{\partial t} + v_1 \frac{\partial \rho}{\partial x_1} + v_2 \frac{\partial \rho}{\partial x_2} + v_3 \frac{\partial \rho}{\partial x_3} = 0$$

Note, for incompressible material

$$\text{div}\vec{V} = 0$$

or

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0$$
3.16 Compatibility Conditions

Recall $\bar{u} = (u_1, u_2, u_3)$ has the three displacement components, but $\bar{E}$ strain tensor has six components $E_{11}, E_{22}, E_{22}, E_{12}, E_{13}, E_{23}$. If $\bar{u}$ is known and continuous $\bar{E}$ exists and is unique. However, given a set of $\bar{E}$, displacement field $\bar{u}$ need not exist, because there are 6 equations and 3 unknowns.

Consider $E_{11} = a_2^2$ all other $E_{ij} = 0$.

\[
\frac{\partial u_1}{\partial X_1} = X_2^2
\]

\[u_1 = X_1X_2^2 + f(X_2, X_3)\]

\[E_{22} = 0 \Rightarrow u_2 = g(X_1, X_3)\]

\[E_{12} = 0 \Rightarrow \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} = 0\]

\[2X_1X_2 + \frac{\partial f(X_2, X_3)}{\partial X_2} + \frac{\partial g(X_1, X_3)}{\partial X_1} = 0\]

\[\Rightarrow h(X_1X_2) + f'(X_2, X_3) + g'(X_1, X_3) \neq 0\]

Hence such a displacement field cannot exist.
3.16 (b) Compatibility (Continued)

If $E_{ij}(X_1, X_2, X_3)$ are continuous with continuous second partial derivatives, to have single valued continuous solutions $u_1, u_2, u_3$, we need additional six equations. They are:

$$
\frac{\partial^2 E_{11}}{\partial X_2^2} + \frac{\partial^2 E_{22}}{\partial X_1^2} = 2 \frac{\partial^2 E_{12}}{\partial X_1 \partial X_2} \\
\frac{\partial^2 E_{33}}{\partial X_1^2} + \frac{\partial^2 E_{11}}{\partial X_3^2} = 2 \frac{\partial^2 E_{31}}{\partial X_3 \partial X_1} \\
\frac{\partial^2 E_{33}}{\partial X_2^2} + \frac{\partial^2 E_{22}}{\partial X_3^2} = 2 \frac{\partial^2 E_{23}}{\partial X_2 \partial X_3} \\
\frac{\partial^2 E_{11}}{\partial X_2 \partial X_3} = \frac{\partial}{\partial X_1} \left( -\frac{\partial E_{23}}{\partial X_1} + \frac{\partial E_{31}}{\partial X_2} + \frac{\partial E_{12}}{\partial X_3} \right) \\
\frac{\partial^2 E_{22}}{\partial X_3 \partial X_1} = \frac{\partial}{\partial X_2} \left( -\frac{\partial E_{31}}{\partial X_2} + \frac{\partial E_{12}}{\partial X_3} + \frac{\partial E_{23}}{\partial X_1} \right) \\
\frac{\partial^2 E_{33}}{\partial X_1 \partial X_2} = \frac{\partial}{\partial X_3} \left( -\frac{\partial E_{12}}{\partial X_3} + \frac{\partial E_{23}}{\partial X_1} + \frac{\partial E_{31}}{\partial X_2} \right)
$$

These equations are called equations of compatibility.

Note:

1) If $\bar{u}$ field is given, compatibility is automatically satisfied.

2) If $E_{ij}$ are linear in $X_1, X_2, X_3$, then all second partial derivative vanish and compatibility is satisfied.
Problem 3.56

Given a strain field, to determine if the field satisfies compatibility

\[
[E] = k \begin{bmatrix}
    X_1^2 & X_2^2 + X_3^2 & X_1 X_3 \\
    X_2^2 + X_3^2 & 0 & X_1 \\
    X_1 X_3 & X_1 & X_2^2 \\
\end{bmatrix}
\]

Answer:

\[
\frac{\partial^2 E_{11}}{\partial X_2^2} + \frac{\partial^2 E_{22}}{\partial X_1^2} = 0 + 0 = 0 \quad \text{while} \quad \frac{\partial^2 E_{12}}{\partial X_1 \partial X_2} = 0, \text{ thus Eq. (3.16.7) is satisfied}
\]

\[
\frac{\partial^2 E_{22}}{\partial X_3^2} + \frac{\partial^2 E_{33}}{\partial X_2^2} = 0 + 2 \neq 0 \quad \text{while} \quad \frac{\partial^2 E_{23}}{\partial X_2 \partial X_3} = 0, \text{ thus Eq. (3.16.8) is not satisfied}
\]

Not satisfied, Hence \textbf{incompatible} field.

The same problem arises when rate of deformation \( D_{ij} \) is specified in fluids problems. However, when velocity field \( v_1, v_2, v_3 \) are specified, compatibility is automatically satisfied.

For strain-rate related problems, when using compatibility equations, replace \( \vec{v} \) by \( \vec{u} \) and \( \vec{E} \) by \( \vec{D} \)
3.18 Deformation Gradient

Let us revisit the concept of deformation gradient, which defined the differential of deformed configuration with respect to undeformed one. Recall that \( F = \frac{\partial \bar{x}}{\partial \bar{X}} \) or \( F_{ij} = x_{ij} \). In terms of spatial coordinates \( \bar{x} = \bar{x}(\bar{X}, t) \)

\[
\begin{align*}
d\bar{x} &= \bar{x}(\bar{X} + d\bar{X}, t) - \bar{x}(\bar{X}, t) \\
&= (\nabla \bar{x}) d\bar{X}
\end{align*}
\]

Thus \( d\bar{x} = \tilde{F} d\bar{X} \) where \( \tilde{F} \) is the deformation gradient at \( \bar{X} \).

In indicial notations, \( dx_i = F_{ij} da_j \)

Since \( x_i = X_i + u_i \)

\[
dx_i = (\delta_{ij} + u_{ij}) dX_j
\]

Thus \( F_{ij} = u_{ij} + \delta_{ij} \) or \( \tilde{F} = \nabla \bar{u} + \bar{I} \)