Part C
Tensor Calculus

In this section of the Chapter 2C, we will examine various types of functions and how to differentiate those functions.

• A scalar has magnitude but no direction at a given point. Examples are density, temperature and pressure. However they will be different in different points in space resulting in scalar-valued functions of space. Position vectors in terms of define the space through \( \mathbf{r} = \mathbf{r}(x, y, z) \). Hence density at all points in space is a scalar field \( \bar{r} = \bar{r}(x, y, z) \).

• A vector at a point defines a quantity, e.g. velocity with a specific magnitude and a direction. Thus at a point \( \mathbf{v} = \mathbf{v}(x = 1, y = -2, z = 1/2) \), velocity will be in a given direction, e.g. \( \frac{1}{\sqrt{6}}(\hat{e}_1 + 2\hat{e}_2 - \hat{e}_3) \) with a magnitude of 30 m/sec. Since this can vary at all points in space specified by \( \bar{r} = \bar{r}(x, y, z) \), then this velocity field is a vector field.
A similar argument can be made for higher order tensor fields. Assume that a tensor e.g. (stress at a point $\vec{r} = \vec{r}(x, y, z)$) is given by the tensor

$$
\tilde{T}(\vec{r}) = \begin{bmatrix}
2 & 3x & -4y^2 - 5z \\
3x & z^4 & 5 + x \\
-4y^2 - 5z & 5 + x & 0
\end{bmatrix}
$$

then the stress varies at every point giving different 9 components (6 since it is symmetric), giving the concept of tensor field.
Tensor Valued Function of a Scalar

• Let \( \tilde{T} = \tilde{T}(t) \) be a tensor-valued function of a scalar \( t \). \( t \) can be time, for example. Differentiating with respect to the scalar (time).

\[
\frac{d\tilde{T}}{dt} = \lim_{\Delta t \to 0} \frac{\tilde{T}(t + \Delta t) - \tilde{T}(t)}{\Delta t}
\]

• For example the tensor-valued function can be represented by:

\[
\tilde{T}(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 2t & -t^2 \\
0 & -t^2 & 5t + 3
\end{bmatrix}
\]

• The usual rules of differential calculus hold good for the above \( \tilde{T} = \tilde{T}(t) \)

• Let us suppose that \( \tilde{T} \) and \( \tilde{S} \) be tensors, \( \tilde{a} \) a vector, and \( \tilde{a}(t) \) a scalar-valued function of \( t \).

\[
\frac{d}{dt}(\tilde{T} + \tilde{S}) = \frac{d\tilde{T}}{dt} + \frac{d\tilde{S}}{dt} \quad \frac{d}{dt}(\alpha(t)\tilde{T}) = \frac{d\alpha}{dt}\tilde{T} + \frac{d\tilde{T}}{dt}\alpha
\]

\[
\frac{d}{dt}(\tilde{T}\tilde{a}) = \frac{d\tilde{T}}{dt}\tilde{a} + \frac{d\tilde{a}}{dt}\tilde{T} \quad \frac{d}{dt}(\tilde{T}^T) = \left(\frac{d\tilde{T}}{dt}\right)^T
\]
Tensor Valued Function of a Scalar

Consider \( \frac{d\tilde{T}}{dt} \), We can show that

\[
\left( \frac{d\tilde{T}}{dt} \right)_{ij} = \frac{d\tilde{T}_{ij}}{dt}
\]

The new tensor can be obtained by differentiating the individual components. In the previous example,

\[
\tilde{T}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2t & -t^2 \\ 0 & -t^2 & 5t + 3 \end{bmatrix}
\]

Thus

\[
\left( \frac{d\tilde{T}}{dt} \right)_{ij} = \frac{d\tilde{T}_{ij}}{dt}
\]

Note that in example problem 2C1.2—if \( \tilde{Q}(t) \) is an orthogonal tensor, then \( \tilde{Q} \tilde{Q}^T \) is an anti-symmetric tensor. Prove it as an exercise.
Example Problem 2

If R(t) is a time dependant rotation tensor, \( \tilde{r}_0 \) is transformed into r(t) by
\[
\tilde{r}(t) = \tilde{R}(t) \cdot \tilde{r}_0
\]

\[
\frac{d\tilde{r}}{dt} = \dot{\tilde{r}} = \tilde{\omega} \times \tilde{r}
\]

where \( \tilde{\omega} \) is the dual vector of \( \tilde{R} \tilde{R}^T \)

Proof:
\[
\frac{d\tilde{r}}{dt} = \frac{d\tilde{R}}{dt} \tilde{r}_0 = \dot{\tilde{R}} \tilde{R}^T \tilde{r} = \tilde{\omega} \times \tilde{r}
\]

Recall that \( \tilde{\omega} \) is the dual vector of \( \tilde{R} \tilde{R}^T \)

From the well known equation in rigid body kinematics, we can identify \( \omega \) as the angular velocity of the body.
2C2 Scalar Field, Gradient of a Scalar Function

Let $\phi(\vec{r})$ be a scalar valued function of $\vec{r}$. $\phi(\vec{r})$ can be density, temperature at $\vec{r}$;

Let us define $\vec{\nabla} = \left( \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right)$

We note that the gradient of $\phi(\vec{r})$ is a vector field $\text{grad}(\phi) = \vec{\nabla}(\phi)$

If we like to find the differential $d\phi$, then

$$d\phi = \phi(\vec{r} + d\vec{r}) - \phi(\vec{r}) \equiv \vec{\nabla}\phi \cdot d\vec{r}$$

Thus $d\phi$ is obtained as a dot product of $\text{grad}(\phi)$ and $d\vec{r}$
Scalar Field, Gradient of a Scalar Function

The gradient by definition is given by,

\[ \vec{\nabla} \phi = \left( \frac{\partial}{\partial x_1} \hat{e}_1 + \frac{\partial}{\partial x_2} \hat{e}_2 + \frac{\partial}{\partial x_3} \hat{e}_3 \right) \]

\[ \vec{\nabla} \phi \cdot dr = 0 \quad \text{for } dr \text{ on iso-} \phi \]

\[ \vec{\nabla} \phi \cdot dr \] is maximum when \( dr \) is normal to iso-\( \phi \), or parallel to \( \vec{\nabla} \phi \)
Example Problem

\[ \phi = x_1 x_2 + x_3 \]

Find \( \hat{n} \) normal to const \( \phi \) at \( \vec{r} = (2,1,0) \)

\[ \nabla \phi = \frac{\partial \phi}{\partial x_1} \hat{e}_1 + \frac{\partial \phi}{\partial x_2} \hat{e}_2 + \frac{\partial \phi}{\partial x_3} \hat{e}_3 = x_2 \hat{e}_1 + x_1 \hat{e}_2 + \hat{e}_3 \]

At (2,1,0)

\[ \begin{cases} 
\nabla \phi = \hat{e}_1 + 2\hat{e}_2 + \hat{e}_3 \\
\hat{n} = \frac{1}{\sqrt{6}} (\hat{e}_1 + 2\hat{e}_2 + \hat{e}_3) 
\end{cases} \]
2C3 Vector Field, Gradient of a Vector Field

• Let \( \vec{v}(\vec{r}) \) be a vector valued function of a displacement or velocity field.

• \( \text{grad}(\vec{v}(\vec{r})) \) is a tensor field \( \nabla \vec{v} \)

\[
d\vec{v} = \vec{v}(\vec{r} + d\vec{r}) - \vec{v}(\vec{r}) \equiv (\nabla \vec{v})d\vec{r}
\]

\[
(\nabla \vec{v})_{ij} = \hat{e}_i \cdot (\nabla \vec{v})\hat{e}_j = \hat{e}_i \cdot \frac{\partial \vec{v}}{\partial x_j} = \frac{\partial}{\partial x_j} (\hat{e}_i \vec{v}) = \frac{\partial v_i}{\partial x_j}
\]

\[
[\nabla \vec{v}] = \begin{bmatrix}
\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\
\frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\
\frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3}
\end{bmatrix}
\]

\[
\frac{\partial v_i}{\partial x_j} \equiv v_{i,j}
\]

• Define \( \frac{\partial v_i}{\partial x_j} \)

• Note that \( ,j \) indicates differentiation in the \( x_j^{th} \) direction.

Also \( \frac{\partial x_i}{\partial x_j} = x_{i,j} = \delta_{i,j} \)
2C4(a) Divergence of a Vector Field and Scalar Field

Let \( \mathbf{\nu}(\mathbf{r}) \) be a vector field. \( \text{div} \mathbf{\nu}(\mathbf{r}) = tr(\nabla \mathbf{\nu}) \)

Recall diagonal element, \( \frac{\partial v_1}{\partial x_1}, \frac{\partial v_2}{\partial x_2}, \frac{\partial v_3}{\partial x_3} \)

\[
\text{div}(\mathbf{\nu}) = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{\partial v_m}{\partial x_m} = v_{m,m}
\]

Note that the divergence of a vector field reduces the order of the vector resulting in a scalar field
2C4(b) Divergence of a Tensor Field

Let \( \tilde{T}(\tilde{r}) \) be a tensor field. \( \text{div}[\tilde{T}(\tilde{r})] \) is a vector field. Here again, note that the divergence reduces the order of tensor yielding vector field.

\[
\text{div}(\tilde{T}) = \frac{\partial T_{im}}{\partial x_m} \hat{e}_i \Rightarrow \frac{\partial T_{im}}{\partial x_m} = T_{im,m}
\]

Example:

If \( \tilde{a} \) and \( \tilde{b} \) are two vectors, and \( \tilde{b} = \alpha \tilde{a} \), show \( \text{div}(\tilde{b}) = \alpha \text{div}(\tilde{a}) + (\nabla \alpha) \tilde{a} \)

\[
\text{div}(\tilde{b}) = \frac{\partial b_i}{\partial x_i} = \alpha \frac{\partial a_i}{\partial x_i} + \frac{\partial \alpha}{\partial x_i} a_i = \alpha \text{div}(\tilde{a}) + (\nabla \alpha) \tilde{a}
\]
2C5 Curl of a Vector Field: - Vector Field

Let $\vec{v}(\vec{r})$ be a vector field. Let $\vec{t}^A$ be a dual vector of $(\nabla \vec{v})^A$

Curl of $\vec{v} = \nabla \times \vec{v} = 2\vec{t}^A$

$$[\nabla \vec{v}]^A = \begin{bmatrix} 0 & \frac{1}{2}(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}) \\ \frac{1}{2}(v_{1,2} - v_{2,1}) & 0 & \frac{1}{2}(v_{2,3} - v_{3,2}) \\ \frac{1}{2}(v_{1,3} - v_{3,1}) & \frac{1}{2}(v_{2,3} - v_{3,2}) & 0 \end{bmatrix}$$

Thus the curl of $\vec{v}(\vec{r})$ is given by

$$\text{curl}(\vec{v}) = 2\vec{t}^A = (v_{3,2} - v_{2,3})\hat{e}_1 + (v_{1,3} - v_{3,1})\hat{e}_2 + (v_{2,1} - v_{1,2})\hat{e}_3$$
Example 2C4

Consider a temperature field given by \( \theta = 3xy \)

(a) Find the heat flux at the point A (1,1,1) if \( q = -k\nabla \theta \)

(b) Find the heat flux at the same point as part (a) if \( q = -K\nabla \theta \), where

\[
[K] = \begin{bmatrix}
k & 0 & 0 \\
0 & 2k & 0 \\
0 & 0 & 3k \\
\end{bmatrix}
\]

Solution:

\[\nabla \theta = 3y e_1 + 3x e_2 \rightarrow (\nabla \theta)_A = 3e_1 + 3e_2\]

(a) \( q = -3k(e_1 + e_2) \)

(b) \[
[q] = \begin{bmatrix}
k & 0 & 0 \\
0 & 2k & 0 \\
0 & 0 & 3k \\
\end{bmatrix} \begin{bmatrix}
3 \\
3k \\
0 \\
\end{bmatrix} = \begin{bmatrix}
3k \\
6k \\
0 \\
\end{bmatrix} \rightarrow q = -3ke_1 - 6ke_2\]