New State-Space Stability Multiplier Tests for Robustness Analysis of δ- and q-Domain Systems

By

Emmanuel G. Collins, Jr.
Department of Mechanical Engineering
Florida A&M – Florida State University
Tallahassee, FL 32310-6046
Tel: (850) 487-8673 FAX: (850) 487-6337
Ecollins@eng.fsu.edu

Wassim M. Haddad
School of Aerospace Engineering
Georgia Institute of Technology
Atlanta, GA 30332-0150
Tel: (404) 894-1078 FAX: (404) 894-2760
wm.haddad@aerospace.gatech.edu

Vijaya-Sekhar Chellaboina
School of Aerospace Engineering
Georgia Institute of Technology
Atlanta, GA 30332-0150
Tel: (404) 894-3474 FAX: (404) 894-3474
Vcl9@cad.gatech.edu

Tinglun Song
Altair Engineering, Inc.
1757 Maplelawn Dr.
Troy, MI 48084
Tel: (248) 614-2400
tinglun@altair.com

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vc19@cad.gatech.edu

Tinglun Song
Altair Engineering, Inc.
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Troy, MI 48084
Tel: (248) 614-2400
tinglun@altair.com

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Abstract

Developments in mixed (i.e., real and complex) structured singular value theory and the associated multiplier-based analysis results have greatly reduced the conservatism in the analysis of systems with mixed uncertainty. However, most of these advances have focused on continuous-time systems, although in practice it is more natural to consider discrete-time systems. Hence, this paper presents multiplier-based robustness tests for the analysis of discrete-time systems. A key objective of this research was to develop discrete-time multiplier tests that reduce to established continuous-time multiplier tests as the sample period approaches zero. This goal led to the use of the delta operator (δ) to represent the discrete-time systems which also has the benefit of avoiding the inherent numerical ill-conditioning at small sample periods resulting from the use of the standard forward-shift operator (q). After presenting a general, multiplier-based, frequency-domain robustness test, it is shown that when only rational multipliers are considered, it is sufficient to consider only a certain class of polynomial multipliers, hence providing a nontrivial extension of a key result of Safonov and Chiang to discrete-time systems. The results are then specialized to δ-domain Popov-type multipliers which leads to the establishment of several new multipliers. For completeness analogous q-domain multipliers and tests are also stated. The results are illustrated with a numerical example.

Keywords: Robustness analysis, discrete-time, δ-domain, fixed-structure multipliers

Running Title: Robustness Analysis of Discrete-Time Systems

1. Introduction

The development of mixed (i.e., real and complex) structured singular value analysis and related multiplier-based analysis results [7, 10, 11] have made a significant impact on the ability of engineers to analyze and design controllers for uncertain systems in the presence of mixed (i.e., real and complex) uncertainty. These results are very powerful and should eventually make a large impact on control engineering practice. However, most of these results have focused on the analysis and synthesis of continuous-time systems. Nevertheless, it is the overwhelming trend to implement controllers digitally.

The focus of this paper is to further advance multiplier-based robustness analysis for discrete-time systems. A key objective is the development of robustness tests that reduce to established continuous-time tests as the sample period approaches zero. This goal led to the representation of discrete-time systems using the delta operator popularized by Goodwin and Middleton [6]. A delta representation of a system (either transfer function or state-space description) has the salient property that the state-space description converges to the underlying continuous-time system as the sample period decreases and hence delta representations are particularly useful for connecting continuous-time and discrete-time results.

Another advantage of delta representations is that they avoid the inherent numerical ill-conditioning inherent in the use of the forward-shift representation at small sample periods. To illustrate this, consider an uncertain, linear, continuous-time system,

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t),$$

where A and B denote the nominal system matrices, and ΔA and ΔB denote uncertain perturbations and T is the sample period . Then, if $u(t)$ is processed via a zero-order-hold, the corresponding forward-shift, discrete-time representation is given by

$$\mathrm{q}x(t) = A_{\mathrm{q}}(\Delta)x(t) + B_{\mathrm{q}}(\Delta)u(t),$$

where

$$A_{\mathrm{q}}(\Delta) \triangleq e^{(A+\Delta A)T}, \quad B_{\mathrm{q}}(\Delta) \triangleq \int_0^T e^{(A+\Delta A)(T-\tau)}(B + \Delta B)\mathrm{d}\tau.$$

A problem with this representation is that regardless of the underlying continuous-time system

$$\lim_{T\to 0} A_{\mathrm{q}}(\Delta) = \lim_{T\to 0} A_{\mathrm{q}}(0) = I, \quad \lim_{T\to 0} B_{\mathrm{q}}(\Delta) = \lim_{T\to 0} B_{\mathrm{q}}(0) = 0.$$

Hence, as the sampling period decreases, the forward-shift representation of a discrete-time system loses its ability to "recognize" the system uncertainty.

Alternatively, consider the discrete-time representation based on the delta-operator [6] defined by

$$\delta(T) \triangleq \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\cdot) & : \quad T = 0, \\ (\mathrm{q}-1)/T : & \quad T > 0. \end{cases}$$

A discrete-time representation of the sampled continuous-time system is given by

$$\delta x(t) = A_\delta(\Delta)x(t) + B_\delta(\Delta)u(t),$$

where

$$\delta(x(t)) = \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(x(t)) & : \quad T = 0, \\ (x(t+T) - x(t))/T : & \quad T > 0, \end{cases}$$

$$A_\delta(\Delta) \triangleq (A_\mathrm{q}(\Delta) - I)/T = (A + \Delta A)\Psi, \quad B_\delta(\Delta) \triangleq B_\mathrm{q}(\Delta)/T = \Psi[B + \Delta B],$$

and

$$\Psi \triangleq I + \frac{(A + \Delta A)T}{2!} + \frac{(A + \Delta A)^2 T^2}{3!} + \ldots.$$

In this case,

$$\lim_{T\to 0} A_\delta(\Delta) = A + \Delta A, \quad \lim_{T\to 0} B_\delta(\Delta) = B + \Delta B.$$

Hence, in contrast to the standard forward-shift representation, the delta representation converges to the underlying uncertain continuous-time system as the sample period decreases.

The developments here focus on delta representations of discrete-time systems and begin by presenting a frequency-domain robustness analysis test involving frequency-dependent multipliers. This test involves strictly generalized positive real and strictly positive real conditions. In order to develop state space robustness tests, emphasis is placed on the development of state-space linear matrix inequality (LMI) and Riccati equation tests for the positive real and generalized positive real conditions.

It is then seen that if the multipliers are restricted to be rational, it is sufficient to consider only a certain class of polynomial multipliers, hence providing a nontrivial extension of a key result of [11] to discrete-time systems and allowing LMI robustness tests to be developed without choosing a basis for the multipliers, which can lead to significant conservatism in the robustness analysis (However, a result from [1] shows that if the multipliers are a linear combination of largely arbitrary basis functions,then relatively nonconservative results can be achieved, provided the basis is sufficiently rich). Unlike the continuous-time results, the multipliers are polynomials in a function of the δ-domain variable γ (or the q-domain variable z). It should be noted that these multiplier constructions differ significantly from those given in [13] when both are simultaneously viewed in either the q-domain or the δ-domain [1]. In fact, when viewed in the δ-domain (as is done here), unlike the multipliers in [13], the multipliers here collapse to the continuous-time multipliers of [11] as $T \to 0$. Two of these multipliers exactly recover the continuous-time Popov multiplier as $T \to 0$. This illustrates that these discrete-time results are not specializations of the continuous-time results. In fact, the converse is true.

Focus is next placed on the development of δ-domain Popov-type multipliers, which because of their simplicity, are particularly useful for robust control law design. This leads to the development of several new Popov-type multipliers that have no counterparts in the existing discrete-time literature.

1.1. Notation and Definitions

The variable γ is the delta transform variable and is the δ-domain analog to the Laplace transform variable s and the Z-transform variable z. (Note that $\gamma = \frac{z-1}{T}$.) The frequency response of a δ-domain transfer function $G(\gamma)$ is given by $G(\nu(\omega))$, where

$$\nu(\omega) \triangleq \begin{cases} j\omega & : \ T = 0, \\ \frac{e^{j\omega T}-1}{T} & : \ T > 0. \end{cases} \tag{1}$$

Let $\mathcal{R}$ and $\mathcal{C}$ denote the real and complex numbers, $\mathcal{R}^{m\times m}$ and $\mathcal{C}^{m\times m}$ the real and complex $m\times m$ matrices, let $(\cdot)^{\mathrm{T}}$ and $(\cdot)^*$ denote transpose and complex conjugate transpose, let "Re" and "Im" denote real and imaginary parts, and let I_n or I denote the $n\times n$ identity matrix. Furthermore we write $\sigma_{\max}(\cdot)$ for the maximum singular value and $M \geq 0$ (resp., $M > 0$) to denote the fact that the Hermitian matrix M is nonnegative (resp., positive) definite. Finally, the Hermitian and skew-Hermitian parts of an arbitrary complex square matrix G are defined by He $G \triangleq \frac{1}{2}(G+G^*)$ and Sh $G \triangleq \frac{1}{2}(G-G^*)$, respectively.

Next, we establish certain key definitions used later in the paper. (Subsequently, the variable "ξ" is used to represent either "z" or "γ".) Let $n(\xi)$ and $d(\xi)$ be polynomials in ξ with real coefficients. A function $g(\xi)$ of the form $g(\xi) = n(\xi)/d(\xi)$ is called a *real rational function*. The function $g(\xi)$ is called *proper* (resp., *strictly proper*) if deg $n(\xi) \leq$ deg $d(\xi)$ (resp., deg $n(\xi) <$ deg $d(\xi)$), where "deg" denotes the degree of the respective polynomials.

[1]Here and elsewhere, "q-domain" is used to refer to both state-space results based on the forward-shift operator and the corresponding transfer function results using the variable z. Likewise, "δ-domain" is used to refer to both the state-space results based on the δ-operator and the corresponding transfer function results using the variable γ.

$G(\gamma)$ is an *asymptotically stable transfer function* if each of its poles are in the open circular region Ω_T with center at $(-\frac{1}{T}, 0)$ and radius $\frac{1}{T}$ given by

$$\Omega_T \triangleq \{\gamma \in \mathcal{C} : \ 2\mathrm{Re}(\gamma) + T|\gamma|^2 < 0\}.$$

Similarly, $G(z)$ is *asymptotically stable* if its poles lie in the open unit disk. $G(\gamma)$ is a *Lyapunov stable transfer function* if each of its poles is in Ω_T with semisimple poles on $\partial\Omega_T \triangleq \{\gamma \in \mathcal{C} : \ 2\mathrm{Re}(\gamma) + T|\gamma|^2 = 0\}$. Similarly, $G(z)$ is a *Lyapunov stable transfer function* if its poles are in the unit disk with semisimple poles on the unit circle.

A square transfer function $G(z)$ is called *positive real* (resp., *generalized positive real*) if $G(z)$ is Lyapunov stable and He $G(z)$ is nonnegative definite for $|z| > 1$ (resp., He $G(e^{j\theta})$ is nonnegative definite for all $\theta \in [0, 2\pi]$ such that $e^{j\theta}$ is not a pole of $G(z)$). A square transfer function $G(z)$ is called *strictly positive real* (resp., *strictly generalized positive real*) if $G(z)$ is asymptotically stable and He $G(e^{j\theta})$ is positive definite for $\theta \in [0, 2\pi]$ (resp., $G(z)$ has no pole on the unit circle and He $G(e^{j\theta})$ is positive definite for all $\theta \in [0, 2\pi]$). Finally, we define the parahermitian conjugate of $G(z)$ by $G^{\sim}(z) \triangleq G^{\mathrm{T}}(1/z)$. Note that $G^{\sim}(e^{j\theta}) = G^{*}(e^{j\theta})$. The corresponding definitions for δ-domain transfer functions can be established by replacing $|z| > 1$, $\theta \in [0, 2\pi]$, and $e^{j\theta}$) with $\gamma \notin \Omega_T \cup \partial\Omega_T$, $\omega \in [-\frac{\pi}{T}, \frac{\pi}{T}]$, and $\nu(\omega)$ respectively.

1.2. Paper Organization

Section 2 presents the general framework for multiplier-based, robustness analysis for δ-domain, discrete-time systems. The main theorem involves strictly generalized positive real and strictly positive real conditions. Hence, to provide the framework for developing state space robustness tests, Section 3 provides state space characterizations of these positive realities. These results are then used to develop LMI robustness tests. Section 4 develops LMI and Riccati equation robustness tests for the special case of δ-domain Popov-type multipliers. Section 5 presents a numerical example. Finally, Section 6 presents conclusions.

2. Multiplier Methods in the Robustness Analysis of Delta-Domain Systems

In this section we consider the standard uncertainty feedback configuration given in Figure 1, where $G(\gamma) \in \mathcal{C}^{m\times m}$ is asymptotically stable and $G(\gamma) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. It is assumed that the uncertainty $\Delta \in \mathcal{C}^{m\times m}$ belongs to the block-diagonal uncertainty set

$$\mathbf{\Delta}_{\mathrm{bs}} \triangleq \{\Delta = \text{block-diag}(\Delta_1, ..., \Delta_p) : \Delta_i \in \mathcal{I}_i,\ i = 1, ..., p\}, \tag{2}$$

where $\mathcal{I}_i \subseteq \mathcal{C}^{k_i\times k_i}$, $\sum_{i=1}^{p} k_i = m$, denotes the internal structure of the uncertainty block Δ_i, $i \in [1, \ldots, p]$. For example $\mathcal{I}_i$, $i \in [1, \ldots, p]$, may be given by any of the following five sets:

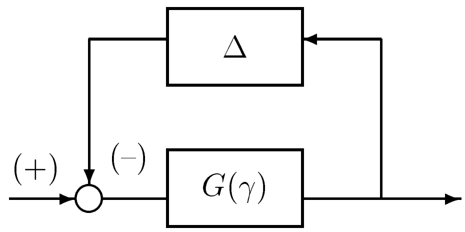

Figure 1: Standard Uncertainty Feedback Configuration

$$\mathbf{\Delta}^{\mathrm{I}} \triangleq \mathcal{C}^{k_i\times k_i}, \tag{3}$$

$$\mathbf{\Delta}^{\mathrm{II}} \triangleq \{\Delta_i = \delta_i I_{k_i} : \delta_i \in \mathcal{C}\}, \tag{4}$$

$$\mathbf{\Delta}^{\mathrm{III}} \triangleq \{\Delta_i = \delta_i I_{k_i} : \delta_i \in \mathcal{R}\}, \tag{5}$$

$$\mathbf{\Delta}^{\mathrm{IV}} \triangleq \{\Delta_i \in \mathcal{R}^{k_i\times k_i} : \Delta_i = \Delta_i^{\mathrm{T}}\}, \tag{6}$$

$$\mathbf{\Delta}^{\mathrm{V}} \triangleq \{\Delta_i = \begin{bmatrix} -\delta_\nu & \delta_\omega \\ -\delta_\omega & -\delta_\nu \end{bmatrix} : \delta_\nu, \delta_\omega \in \mathcal{R}\}. \tag{7}$$

Note that $\mathbf{\Delta}^{\mathrm{I}}, \mathbf{\Delta}^{\mathrm{II}}$, and $\mathbf{\Delta}^{\mathrm{III}}$ are standard in the literature, corresponding respectively to complex matrix block uncertainty, repeated complex scalar uncertainty, and repeated real scalar uncertainty. $\mathbf{\Delta}^{\mathrm{IV}}$ is symmetric, real matrix block uncertainty, while $\mathbf{\Delta}^{\mathrm{V}}$ can be used to describe uncertainty in the imaginary and real eigenvalues of a system in which A is represented in real normal form. If the uncertainty is of the form described by $\mathbf{\Delta}^{\mathrm{IV}}$ or $\mathbf{\Delta}^{\mathrm{V}}$, it is possible to represent the uncertainty by $\mathbf{\Delta}^{\mathrm{III}}$. However, as discussed in [8] this reformulation leads to increased conservatism and numerical complexity. The ensuing discussion is not restricted to these forms of uncertainty, but they are important special cases and will be used to provide concrete illustrations of the subsequent concepts.

Next, define the subset $\mathbf{\Delta} \subseteq \mathbf{\Delta}_{\mathrm{bs}}$ consisting of sector-bounded matrices

$$\mathbf{\Delta} \triangleq \{\Delta \in \mathbf{\Delta}_{\mathrm{bs}} : 2(\Delta - M_1)^*(M_2 - M_1)^{-1}(\Delta - M_1) \leq (\Delta - M_1) + (\Delta - M_1)^*\},$$

where $M_1, M_2 \in \mathbf{\Delta}_{\rm bs}$ are Hermitian matrices such that $M \triangleq M_2 - M_1$ is positive definite. Note that M_1 and M_2 are elements of $\mathbf{\Delta}$. The following lemma characterizes important special cases of $\mathbf{\Delta}$.

Lemma 1. [8]. Let $\Delta \in \mathbf{\Delta}$. Then $\Delta = \Delta^*$ if and only if $M_1 \leq \Delta \leq M_2$. Furthermore, suppose $M_1 = -\eta I$ and $M_2 = \eta I$, where $\eta > 0$. Then,

$$\mathbf{\Delta} = \mathbf{\Delta}_\eta \triangleq \{\Delta \in \mathbf{\Delta}_{\rm bs} : \sigma_{\max}(\Delta) \leq \eta\}. \tag{8}$$

Remark 1. Note that for nonrepeated scalar uncertainty, Δ is real and diagonal and hence satisfies $M_1 \leq \Delta \leq M_2$. Furthermore, note that $\mathbf{\Delta}_\eta$ is the standard norm-bounded uncertainty set used to define the structured singular value.

To state the multivariable absolute stability criterion for δ-domain systems with $\Delta \in \mathbf{\Delta}$ we define, for $i \in [1, \ldots, p]$, the sets of Hermitian, frequency-dependent, scaling matrix functions by

$$\begin{aligned}\mathcal{D}_i \triangleq\ & \{D_i : \nu([-\tfrac{\pi}{T}, \tfrac{\pi}{T}]) \to \mathcal{C}^{k_i \times k_i} : D_i(\nu(\omega)) \geq 0, \\ & D_i(\nu(\omega))\Delta_i = \Delta_i D_i(\nu(\omega)),\ \omega \in [-\tfrac{\pi}{T}, \tfrac{\pi}{T}],\ \Delta_i \in \mathcal{I}_i\},\end{aligned} \tag{9}$$

$$\begin{aligned}\mathcal{N}_i \triangleq\ & \{N_i : \nu([-\tfrac{\pi}{T}, \tfrac{\pi}{T}]) \to \mathcal{C}^{k_i \times k_i} : N_i(\nu(\omega)) = {N_i}^*(\nu(\omega)), \\ & N_i(\nu(\omega))\Delta_i = {\Delta_i}^* N_i(\nu(\omega)),\ \omega \in [-\tfrac{\pi}{T}, \tfrac{\pi}{T}],\ \Delta_i \in \mathcal{I}_i\}.\end{aligned} \tag{10}$$

Furthermore, define the sets $\mathcal{M}_i$, $i \in [1, \ldots, p]$, and $\mathcal{M}$ of frequency-dependent multiplier matrix functions by

$$\begin{aligned}\mathcal{M}_i \triangleq \{M_i(\nu(\omega)) = D_i(\nu(\omega)) + Q_i(\nu(\omega)) : D_i(\nu(\omega)) \in \mathcal{D}_i, \quad Q_i(\nu(\omega)) = jN_i(\nu(\omega)), \\ N_i(\nu(\omega)) \in \mathcal{N}_i\},\end{aligned} \tag{11}$$

$$\mathcal{M} \triangleq \{M(\nu(\omega)) \in \mathcal{C}^{m \times m} : M(\nu(\omega)) \ = \ \text{block-diag}(M_1(\nu(\omega)), ..., M_p(\nu(\omega))), \tag{12}$$

$$M_i(\nu(\omega)) \in \mathcal{M}_i,\ i = 1, ..., p\}. \tag{13}$$

Note that in (11), $D_i(\nu(\omega)) = \text{He}\ M_i(\nu(\omega)) \geq 0$ and $N_i(\nu(\omega)) = -j\text{Sh}\ M_i(\nu(\omega))$. Furthermore, $M(\nu(\omega)) \in \mathcal{M}$ satisfies

$$\text{He}\ M(\nu(\omega)) \geq 0, \ \ \omega \in [-\tfrac{\pi}{T}, \tfrac{\pi}{T}]. \tag{14}$$

Theorem 1. Suppose that $G(\gamma)[I+M_1G(\gamma)]^{-1}$ is asymptotically stable and $\det(I+G(-\frac{2}{T})\Delta)\neq 0$ for all $\Delta\in\mathbf{\Delta}$. If there exists $M(\nu(\omega))\in\mathcal{M}$ such that

$$\mathrm{He}[M(\nu(\omega))\hat{G}(\nu(\omega))]>0,\ \omega\in(-\tfrac{\pi}{T},\tfrac{\pi}{T}), \tag{15}$$

where

$$\hat{G}(\gamma)\triangleq[I+M_2G(\gamma)][I+M_1G(\gamma)]^{-1}, \tag{16}$$

then the negative feedback interconnection of $G(\gamma)$ and Δ is asymptotically stable (or, equivalently, $\det(I+G(\nu(\omega))\Delta)\neq 0,\ \omega\in[-\frac{\pi}{T},\frac{\pi}{T}]$) for all $\Delta\in\mathbf{\Delta}$.

Proof. The proof is obtainable by bilinear transformation of the corresponding continuous-time result [8]. An alternative direct proof is given in [12]. □

Remark 2. As in the continuous-time case [8], the results of Theorem 1 may be used to develop frequency-dependent robustness tests for δ-domain, discrete-time systems.

Equivalent q-Domain Results

Consider the standard uncertainty feedback configuration given in Figure 1 with "$G(\gamma)$" replaced by "$G(z)$" where $G(z)\in\mathcal{C}^{m\times m}$ is asymptotically stable. The set $\mathcal{M}$ of frequency-dependent multipliers is defined by (9)-(13) with "$\nu(\omega)$" replaced by "$e^{j\theta}$" and "$\omega\in[-\frac{\pi}{T},\frac{\pi}{T}]$" replaced by "$\theta\in[0,2\pi]$". Furthermore, $M(e^{j\theta})\in\mathcal{M}$ satisfies

$$\mathrm{He}\ M(e^{j\theta})\geq 0,\ \ \theta\in[0,2\pi]. \tag{17}$$

The following theorem provides a frequency domain robust stability measure.

Theorem 2. Suppose that $G(z)[I+M_1G(z)]^{-1}$ is asymptotically stable and $\det(I+G(-1)\Delta)\neq 0$ for all $\Delta\in\mathbf{\Delta}$. If there exists $M(e^{j\theta})\in\mathcal{M}$ such that

$$\mathrm{He}[M(e^{j\theta})\hat{G}(e^{j\theta})]>0,\ \theta\in[0,2\pi], \tag{18}$$

where

$$\hat{G}(z)\triangleq[I+M_2G(z)][I+M_1G(z)]^{-1}, \tag{19}$$

then the negative feedback interconnection of $G(z)$ and Δ is asymptotically stable (or, equivalently, $\det(I+G(e^{j\theta})\Delta)\neq 0,\ \theta\in[0,2\pi]$) for all $\Delta\in\mathbf{\Delta}$.

3. LMI State Space Robustness Tests Using the Delta Operator

Theorem 1 and (14) characterize robustness in terms of the strictly generalized positive real and strictly positive real conditions. Hence, to implement the robustness test of Theorem 1 using state space computations requires state space characterizations of strictly generalized positive real and strictly positive real transfer functions.

Proposition 1. Let $G(\gamma)$ be square with $G(\gamma) \sim \left[\begin{array}{c|c} A_\delta & B_\delta \\ \hline C & D \end{array}\right]$, where (A_δ, B_δ) is controllable and A_δ has no eigenvalues on Ω_T. Then the following statements are equivalent:

1. $G(\gamma)$ is strictly generalized positive real.

2. He $G(\nu(\omega)) > 0, \;\; \omega[-\frac{\pi}{T}, \frac{\pi}{T}]$.

3. There exists $\epsilon > 0$ such that He $G(\nu(\omega)) \geq \epsilon I, \quad \omega \in [-\frac{\pi}{T}, \frac{\pi}{T}]$.

4. There exist $P = P^{\mathrm{T}}$ and $\epsilon > 0$ such that
$$\left[\begin{array}{cc} -TA_\delta^{\mathrm{T}} PA_\delta - A_\delta^{\mathrm{T}} P - PA_\delta & -TA_\delta^{\mathrm{T}} PB_\delta - PB_\delta + C^{\mathrm{T}} \\ -TB_\delta^{\mathrm{T}} PA_\delta - B_\delta^{\mathrm{T}} P + C & -TB_\delta^{\mathrm{T}} PB_\delta + (D - \epsilon I) + (D - \epsilon I)^{\mathrm{T}} \end{array}\right] \geq 0. \tag{20}$$

   Furthermore, the following statements hold:

5. (A_δ, C) is observable if and only if P is nonsingular.

6. $G(\gamma)$ is strictly positive real if and only if A_δ is δ-domain asymptotically stable and there exist $P \geq 0$ and $\epsilon > 0$ such that (20) holds.

7. $G(\gamma)$ is strictly positive real with (A_δ, C) observable if and only if there exists $P > 0$ such that
$$\left[\begin{array}{cc} -TA_\delta^{\mathrm{T}} PA_\delta - A_\delta^{\mathrm{T}} P - PA_\delta & -TA_\delta^{\mathrm{T}} PB_\delta - PB_\delta + C^{\mathrm{T}} \\ -TB_\delta^{\mathrm{T}} PA_\delta - B_\delta^{\mathrm{T}} P + C & -TB_\delta^{\mathrm{T}} PB_\delta + D + D^{\mathrm{T}} \end{array}\right] > 0. \tag{21}$$

8. $G(\gamma)$ is strictly positive real with (A_δ, C) observable if and only if there exist $P > 0$ and $R > 0$ such that
$$D + D^{\mathrm{T}} - TB_\delta^{\mathrm{T}} PB_\delta > 0, \tag{22}$$
$$\begin{aligned} 0 = {} & TA_\delta^{\mathrm{T}} PA_\delta + A_\delta^{\mathrm{T}} P + PA_\delta \\ & + [B_\delta^{\mathrm{T}} P(TA_\delta + I) - C]^{\mathrm{T}} (D + D^{\mathrm{T}} - TB_\delta^{\mathrm{T}} PB_\delta)^{-1} [B_\delta^{\mathrm{T}} P(TA_\delta + I) - C] + R. \end{aligned} \tag{23}$$

Proof. The proof is a relatively straightforward generalization of similar q-domain results from [9]. For details of the proof, see [12]. □

Remark 3. The equivalent q-domain results are also given and proved in [12].

Robustness Tests

Propositions 1 show that for δ-domain systems generalized positive reality and strictly generalized positive reality result in LMI tests. Furthermore, the latter condition (under the assumption that $(A_\delta, B_\delta, C, D)$ is a minimal state space realization) is equivalent to the existence of a solution to a Riccati equation. The key in the development of LMI robustness tests is to represent $M(\gamma)$ in the strictly generalized positive real test (15) of Theorem 1 in such a way that its parameters appear linearly within the corresponding state space tests of Proposition 4. For the following result define

$$\psi(\gamma) \triangleq \frac{2\gamma}{2+\gamma T}$$

and note that $\psi^{\sim}(\gamma) = -\psi(\gamma)$.

Lemma 2. Let $n(\gamma)$ and $d(\gamma)$ be real polynomials. Then there exist real polynomials $\hat{n}(\psi(\gamma))$ and $\hat{d}(\psi(\gamma))$ in $\psi(\gamma)$ such that

$$\frac{n(\gamma)}{d(\gamma)} = \frac{\hat{n}(\psi(\gamma))}{\hat{d}(\psi(\gamma))}, \tag{24}$$

and $\max\{\deg(\hat{n}(\gamma)),\ \deg(\hat{d}(\gamma))\} \leq \max\{\deg(n(\gamma)),\ \deg(d(\gamma))\}$.

Proof. First note that $n(\gamma)$ and $d(\gamma)$ can be expressed as real polynomials of the form $\tilde{n}(2+\gamma T)$ and $\tilde{d}(2+\gamma T)$. Now, let $n = \max\{\deg(n(\gamma)),\ \deg(d(\gamma))\}$ and note that $\max\{\deg(\tilde{n}(\gamma)), \deg(\tilde{d}(\gamma))\} = n$. Next, note that

$$\frac{n(\gamma)}{d(\gamma)} = \frac{\tilde{n}(2+\gamma T)}{\tilde{d}(2+\gamma T)} = \frac{(2+\gamma T)^{-n}\tilde{n}(2+\gamma T)}{(2+\gamma T)^{-n}\tilde{d}(2+\gamma T)},$$

which implies that $\frac{n(\gamma)}{d(\gamma)}$ can be expressed as ratio of real polynomials in $(2+\gamma T)^{-1}$. Now the result follows by noting that $(2+\gamma T)^{-1} = \frac{1}{2}(1 - T\psi(\gamma))$. □

Remark 4. It follows from Lemma 2 that given a real-rational matrix function $M(\gamma)$ there exists a real polynomial matrix function $N(\psi(\gamma))$ and a scalar real polynomial $d(\psi(\gamma))$ such that $M(\gamma) = \frac{1}{d(\psi(\gamma))}N(\psi(\gamma))$.

Next, we present a result that *nonconservatively* restricts the multiplier search to real polynomials in $\psi(\gamma)$.

Lemma 3. There exists a matrix function $M_0(\gamma) \in \mathcal{M}$ such that (15) holds with $M(\gamma) = M_0(\gamma)$ if and only if there exists a real polynomial matrix function $M_{\rm p}(\psi(\gamma))$ such that (15) holds with $M(\gamma) = M_{\rm p}(\psi(\gamma))$. Furthermore, if $M_0(\gamma)$ is factored as $M_0(\gamma) = (1/d_0(\gamma))N_0(\gamma)$, where $N_0(\gamma)$ is a real polynomial matrix function and $d_0(\gamma)$ is a real scalar polynomial, then $M_{\rm p}(\psi(\gamma))$ can be chosen such that $\deg(M_{\rm p}(\psi(\gamma))) \leq \deg(N_0(\gamma)) + \deg(d_0(\gamma))$.

Proof. Given a matrix function $M_0(\gamma)$ it follows from Lemma 2 that there exists a real-rational matrix function $\hat{M}_0(\psi(\gamma))$ such that $M_0(\gamma) = \hat{M}_0(\psi(\gamma))$. Now the result follows from Lemma 3 of [11]. $\square$

Remark 5. Note that, as $T \to 0$ Lemma 3 recovers Lemma 3 of [11].

Remark 6. Note that since $\psi(\gamma)$ is proper it follows that any real polynomial matrix $M_{\rm p}(\psi(\gamma))$ is also proper and hence state space realizable.

Remark 7. As the order of the real polynomial matrix function $M_{\rm p}(\psi(\gamma))$ increases, the conservatism of the analysis result decreases. However, Lemma 3 does not provide guidance for the choice of the order of $M_{\rm p}(\psi(\gamma))$. This must be determined experimentally.

Next we show that, given a fixed-order real polynomial matrix function $M_{\rm p}(\psi(\gamma))$, the state space realization of $M_{\rm p}(\psi(\gamma))\hat{G}(\gamma)$ depends linearly and additively on the multiplier coefficients.

Lemma 4. Let $\hat{G}(\gamma) \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$ and let $M_{\rm p}(\psi(\gamma))$ be given by $M_{\rm p}(\psi(\gamma)) = \sum_{i=0}^{n} M_i \psi^i(\gamma)$ where $M_i \in \mathcal{C}^{m\times m}$. If $M_i\psi^i(\gamma) \sim \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right]$, $i = 1, \ldots, n$, then a state space realization of $M_{\rm p}(\psi(\gamma))\hat{G}(\gamma)$ is given by

$$M_{\rm p}(\psi(\gamma))\hat{G}(\gamma) \sim \left[\begin{array}{cc|c} \hat{A} & 0 & \hat{B} \\ B_{\rm p}\hat{C} & A_{\rm p} & B_{\rm p}\hat{D} \\ \hline D_{\rm p}\hat{C} & C_{\rm p} & D_{\rm p}\hat{D} \end{array}\right], \qquad (25)$$

where

$$A_{\rm p} \triangleq \text{block-diag}(A_0, \ldots, A_n), \quad B_{\rm p} \triangleq [B_0^{\rm T}, \ldots, B_n^{\rm T}]^{\rm T}, \quad C_{\rm p} \triangleq [C_0, \ldots, C_n], \quad D_{\rm p} \triangleq \sum_{i=0}^{n} D_i.$$

Proof. Note that $M_{\mathrm{p}}(\psi(\gamma)) \sim \left[\begin{array}{c|c} A_{\mathrm{p}} & B_{\mathrm{p}} \\ \hline C_{\mathrm{p}} & D_{\mathrm{p}} \end{array}\right]$. Now, the result is immediate by using cascade realizations. □

Next, for convenience define

$$A_{\mathrm{m}} \triangleq \begin{bmatrix} \hat{A} & 0 \\ B_{\mathrm{p}}\hat{C} & A_{\mathrm{p}} \end{bmatrix}, \quad B_{\mathrm{m}} \triangleq \begin{bmatrix} \hat{B} \\ B_{\mathrm{p}}\hat{D} \end{bmatrix}, \quad C_{\mathrm{m}} \triangleq \begin{bmatrix} D_{\mathrm{p}}\hat{C} & C_{\mathrm{p}} \end{bmatrix}, \quad D_{\mathrm{m}} \triangleq D_{\mathrm{p}}\hat{D}. \tag{26}$$

Now, we use Proposition 1 to transform the conditions (14) and (15) to their equivalent LMI tests. However, since A_{p} can have an eigenvalue on $\partial\Omega_T$, we replace A_{p} by $\rho A_{\mathrm{p}} + \frac{\rho-1}{T}I$, where $\rho < 1$ but sufficiently close to 1 so that $\rho A_{\mathrm{p}} + \frac{\rho-1}{T}I$ is asymptotically stable. With this substitution the following result is immediate.

Theorem 3. Suppose $G(\gamma)[I+M_1G(\gamma)]^{-1}$ is asymptotically stable. If there exists $P > 0$ such that

$$\begin{bmatrix} -TA_{\mathrm{a}}^{\mathrm{T}}PA_{\mathrm{a}} - A_{\mathrm{a}}^{\mathrm{T}}P - PA_{\mathrm{a}} & -TA_{\mathrm{a}}^{\mathrm{T}}PB_{\mathrm{a}} - PB_{\mathrm{a}} + C_{\mathrm{a}}^{\mathrm{T}} \\ -TB_{\mathrm{a}}^{\mathrm{T}}PA_{\mathrm{a}} - B_{\mathrm{a}}^{\mathrm{T}}P + C_{\mathrm{a}} & -TB_{\mathrm{a}}^{\mathrm{T}}PB_{\mathrm{a}} + D_{\mathrm{a}} + D_{\mathrm{a}}^{\mathrm{T}} \end{bmatrix} > 0, \tag{27}$$

where

$$A_{\mathrm{a}} \triangleq \text{block-diag}(A_{\mathrm{p}}, A_{\mathrm{m}}), \qquad B_{\mathrm{a}} \triangleq \text{block-diag}(B_{\mathrm{p}}, B_{\mathrm{m}}),$$
$$C_{\mathrm{a}} \triangleq \text{block-diag}(C_{\mathrm{p}}, C_{\mathrm{m}}), \qquad D_{\mathrm{a}} \triangleq \text{block-diag}(D_{\mathrm{p}}, D_{\mathrm{m}}),$$

then conditions (14) and (15) hold. In this case the negative feedback interconnection of $G(\gamma)$ and Δ is asymptotically stable for all $\Delta \in \boldsymbol{\Delta}$.

Equivalent q-Domain Results

For the following result define

$$\psi(z) \triangleq \frac{z-1}{z+1}$$

and note that $\psi^{\sim}(z) = -\psi(z)$.

Lemma 5. Let $n(z)$ and $d(z)$ be real polynomials. Then there exist real polynomials $\hat{n}(\psi(z))$ and $\hat{d}(\psi(z))$ in $\psi(z)$ such that

$$\frac{n(z)}{d(z)} = \frac{\hat{n}(\psi(z))}{\hat{d}(\psi(z))}, \tag{28}$$

and $\max\{\deg(\hat{n}(z)),\ \deg(\hat{d}(z))\} \leq \max\{\deg(n(z)),\ \deg(d(z))\}$.

Remark 8. Note that since $\psi(z)$ is proper and it follows that any real polynomial matrix $M_P(\psi(z))$ is also proper and hence state-space realizable.

Next, we present a result that *nonconservatively* restricts the multiplier search to real polynomials in $\psi(z)$.

Lemma 6. There exists a matrix function $M_0(z) \in \mathcal{M}$ such that (18) holds with $M(z) = M_0(z)$ if and only if there exists a real polynomial matrix function $M_{\rm p}(\psi(z))$ such that (18) holds with $M(z) = M_{\rm p}(\psi(z))$. Furthermore, if $M_0(z)$ is factored as $M_0(z) = (1/d_0(z))N_0(z)$, where $N_0(z)$ is a real polynomial matrix function and $d_0(z)$ is a real scalar polynomial, then $M_{\rm p}(\psi(z))$ can be chosen such that $\deg(M_{\rm p}(\psi(z))) \leq \deg(N_0(z)) + \deg(d_0(z))$.

4. Robustness Analysis Using Delta-Domain Popov-Type Multipliers

In this section we consider several Popov-type multipliers in the δ-domain and present the q-domain versions of these multipliers, of which all but one, the Popov-Tsypkin multiplier, have *not* appeared in the literature. The new multipliers were found because of the close connection between the continuous-time and δ-domain systems. These discrete-time multipliers are *not* specializations of the continuous-time Popov multiplier. In fact, since two of them exactly recover the continuous-time Popov multiplier as $T \to 0$, the converse if true.

The advantage of the Popov-type multipliers is their simplicity which makes them particularly useful for control law design. For simplicity of exposition we constrain $\mathbf{\Delta}_{\rm bs}$ to the set of diagonal uncertainties so that diagonal frequency-dependent multiplier constructions automatically satisfy the compatibility assumptions discussed in Section 2. The corresponding LMI and Riccati equation state-space tests are easily developed using Theorem 1 and Proposition 1. The details are given in [12].

4.1. Popov-Type Multiplier 1

First, we consider the monomial in $\psi(\gamma)$ multiplier given by

$$M(\gamma) = H + N\frac{2\gamma}{2 + \gamma T}, \tag{29}$$

where H and N are real and diagonal with $H > 0$. Note that this multiplier recovers the continuous-time Popov multiplier $H + Ns$ as $T \to 0$. To see that this multiplier is in the class of multipliers for real parametric uncertainty, that is, $M(\gamma) \in \mathcal{M}_i$, $i \in [1, \ldots, p]$, note that

$$\mathrm{He}(H + N\frac{2\gamma}{2+\gamma T})|_{\gamma=\nu(\omega)} = \mathrm{He}(H + \frac{2N}{T} j \tan\frac{\omega T}{2}) = H > 0. \tag{30}$$

Furthermore, if $N \geq 0$ then

$$\mathrm{He}(H + N\frac{2\gamma}{2+\gamma T}) = H + 2N\frac{2\mathrm{Re}(\gamma) + T|\gamma|^2}{|2+\gamma T|^2} > 0, \qquad \gamma \notin \Omega_T \cup \partial\Omega_T, \tag{31}$$

which implies that $H + N\frac{2\gamma}{2+\gamma T}$ is a positive real multiplier. This multiplier and each of those given below can also be applied to complex uncertainty by choosing $N = 0$ since $H \in \mathcal{M}_i$, $i \in [1, \ldots, p]$.

Remark 9. The q-domain multiplier corresponding to (29) is given by

$$M(z) = H + N\frac{z-1}{z+1}. \tag{32}$$

4.2. Popov-Type Multiplier 2

Now, we consider the Popov-type multiplier

$$M(\gamma) = H + N\frac{\gamma(2+\gamma T)}{2(1+\gamma T)}, \tag{33}$$

where H and N are real and diagonal and $H > 0$. Once again the multiplier (33) recovers the continuous-time Popov multiplier $H + Ns$ as $T \to 0$. Furthermore, since

$$\mathrm{He}(H + N\frac{\gamma(2+\gamma T)}{2(1+\gamma T)})|_{\gamma=\nu(\omega)} = \mathrm{He}(H + Nj\frac{\sin\omega T}{T}) = H > 0, \tag{34}$$

it follows that $M(\gamma) \in \mathcal{M}_i$, $i \in [1, \ldots, p]$. Note that this multiplier is not positive real. However, since this multiplier is asymptotically stable, the robustness test (15) can be performed using a strictly positive real test with the state-space realizations of $M(\gamma)\hat{G}(\gamma)$.

Remark 10. The q-domain multiplier corresponding to (33) is given by

$$M(z) = H + N\frac{(z+1)(z-1)}{z}. \tag{35}$$

4.3. A Class of Popov-Type Multipliers

Next we consider the class of Popov-type multiplier described by

$$M_a(\gamma) = H + N\frac{a\gamma}{a+\gamma T}, \qquad 1 \le a < 2, \tag{36}$$

where $H > 0$ and $N \ge 0$ are real and diagonal. Note that for $a \in [1, 2)$, $M_a(\gamma) \in \mathcal{M}_i$, $i \in [1, ..., p]$, since

$$\mathrm{He}[H + N\frac{a\gamma}{a+\gamma T}] = H + \frac{aN}{T}\frac{(2-a)(1-\cos\omega T)}{a^2 - 2(a-1)(1-\cos\omega T)} > 0. \tag{37}$$

Note that since we require $N \ge 0$, unlike the Popov-type multiplier given by (33), each of the multipliers in this class do not recover the continuous-time Popov multiplier as $T \to 0$. For $a = 2$, $M_a(\gamma)$ becomes the multiplier given by (29) however $N \ge 0$ is no longer required.

Remark 11. The q-domain multiplier corresponding to (36) is given by

$$M_a(z) = H + N\frac{z-1}{z+(a-1)}, \quad 1 \le a < 2, \tag{38}$$

where $H > 0$ and $N \ge 0$ are real and diagonal.

Remark 12. If $a = 1$, $M_a(z)$ corresponds to the Popov-Tsypkin multiplier in the classical discrete-time absolute stability literature [14].

4.4. Design of Robust Controllers and Estimators

An important application of Popov-type multipliers is the design of fixed-architecture, robust controllers or estimators. See, for example [2, 3, 4] for robust controller design for continuous-time systems using the Popov multiplier. See [5] for robust estimator design for q-domain systems using the Popov-Tsypkin multiplier.

5. Illustrative Numerical Example

To illustrate the δ-domain robustness analysis framework presented in the previous sections, we consider the two mass/spring benchmark system with uncertain stiffness k. For a detailed discussion of this benchmark problem, refer to [2] and the references therein.

The open-loop plant (for $m_1 = m_2 = 1$) is given by

$$\begin{aligned} \dot{x}_{\rm p} &= (A_{\rm p} + \Delta A_{\rm p})x_{\rm p} + B_{\rm p}u, && (39) \\ y &= C_{\rm p}x_{\rm p}, && (40) \\ z_{\rm p} &= E_1 x_{\rm p}, && (41) \end{aligned}$$

where

$$x_{\rm p} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \; A_{\rm p} + \Delta A_{\rm p} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & 0 & 0 \\ k & -k & 0 & 0 \end{bmatrix}, \; B_{\rm p} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C_{\rm p} = E_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}.$$

A robustly stabilizing $4^{\rm th}$ order dynamic controller guaranteeing stability for $0.5 \le k \le 2.0$ is presented in [2] and is given here in state space form as

$$\begin{aligned} \dot{x}_{\rm c} &= A_{\rm c}x_{\rm c} + B_{\rm c}y, && (42) \\ u &= -C_{\rm c}x_{\rm c}, && (43) \end{aligned}$$

where

$$A_{\rm c} = \begin{bmatrix} -15.0 & -186.6 & -1028.1 & -2162.0 \\ 1.0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 \end{bmatrix}, \; B_{\rm c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C_{\rm c} = \begin{bmatrix} 2819.0 & 34.7 & 1745.1 & 386.6 \end{bmatrix}.$$

Using Popov analysis this controller guarantees robust stability for the range $0.334 \le k \le 2.166$ which corresponds to the actual stability margin with $\eta = \eta_{\rm achieved} = 0.916$.

Next we compare the δ-domain and q-domain robust stability tests using multipliers 1, 2, and 3. The robustness tests are formulated as LMI feasibility problems with varying sampling periods. Figure 2 shows the comparison of the allowable uncertainty predictions. It is clear from the figure that as $T \to 0$ the robustness tests for the δ-domain representation continue to track the continuous-time robustness analysis predictions while the q-domain formulation fails to recover the continuous-time robustness boundaries.

It should be noted that less conservative results can be obtained by using higher order polynomial multipliers as discussed in Section 3. In addition, if the system being analyzed is inherently discrete-time (i.e., it is not obtained by sampling a continuous-time system), the δ-domain formulation may not yield better performance than the q-domain formulation.

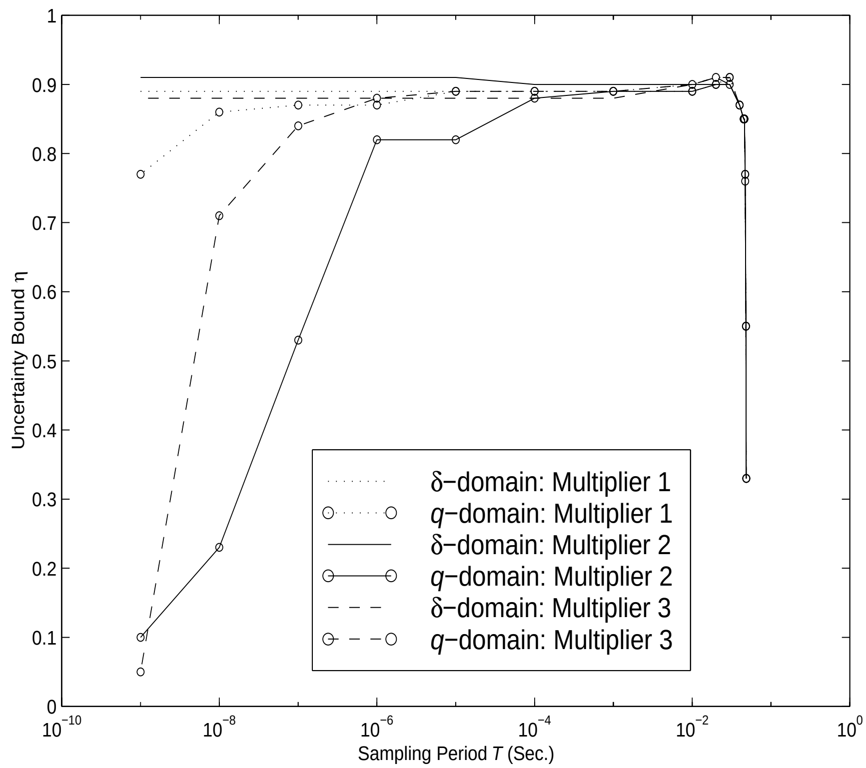

Figure 2: Comparison of the q-domain and δ-domain Robustness Predictions as a Function of Sampling Period T

6. Conclusion

A general theory for the robustness analysis of discrete-time systems using multiplier theory was presented. To make connections to existing continuous-time robustness theory, the developments were primarily in the δ-domain and the parallel q-domain results were subsequently discussed. The foundational frequency-domain robustness test result in terms of generalized positive real conditions is analogous to an existing result for continuous-time systems. To enable the development of state space robustness analysis tests, LMI and Riccati equation characterizations of strictly generalized positive real conditions for discrete-time systems are easily developed. Some of these results are new. A key result was then the nontrivial extension to discrete-time systems of an important result of [11], which allowed the development of state-space LMI robustness tests without taking the conservative step of choosing a basis for the multiplier. Another significant contribution was the development of Popov-type multipliers since the q-domain version of these multipliers (with the exception of the Popov-Tsypkin multiplier) have not appeared in the existing literature. For the Popov-type multipliers, both LMI and Riccati equation robustness tests were developed. These results can be used for fixed-architecture robust controller or estimator design. Robustness analysis using the developed results was illustrated with a numerical example. As expected, the δ-domain analysis exhibited much better numerical robustness at low sample periods than the corresponding q-domain analysis.

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