Robust H_∞ Estimation and Fault Detection of Uncertain Dynamic Systems

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Abstract

This paper uses the Popov-Tsypkin multiplier (which has intimate connections to mixed structured singular value theory) to design robust H_∞ estimators for uncertain, linear discrete-time systems and considers the application of robust H_∞ estimators to robust fault detection. The key to estimator-based, robust fault detection is to generate residuals which are robust against plant uncertainties and external disturbance inputs, which in turn requires the design of robust estimators. The robust H_∞ estimation problem is formulated as a Riccati equation feasibility problem in which a cost function is minimized subject to a Riccati equation constraint. A continuation algorithm that uses quasi-Newton BFGS (the algorithm of Broyden, Fletcher, Goldfab, and Shanno) corrections is developed to solve the minimization problem. The algorithm is initialized with an H_∞ estimator designed for the nominal system. The initializing multiplier matrices are obtained by solving a linear matrix inequality. The robust H_∞ estimator framework is then applied to the robust fault detection of dynamic systems. The results are applied to a simplified longitudinal flight control system. It is shown that the robust fault detection procedure based on the robust H_∞ estimation methodology proposed in this paper can reduce the false alarm rate.

Keywords: robust H_∞ estimation, multiplier theory, mixed structured singular value, robust fault detection

Running Title: Robust H_∞ estimation and fault detection

1. Introduction

Fault detection of dynamic systems using analytical approach has been an active research area in recent years [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?]. Fault detection can be achieved by using either *physical redundancy* or *analytical redundancy* (e.g., estimator-based fault detection) methods. The key step in estimator-based fault detection methods is to generate residuals which are accentuated by faults. These residuals are then compared with some threshold values to determine whether faults have occurred. Logically, the existence of uncertainties and disturbance inputs (i.e., plant disturbances and measurement noise) obscures the effect of faults and is therefore a source of false alarms. In order to reduce false alarm rates and improve fault detection accuracy, the residuals generated should be robust against uncertainties and disturbance inputs. The residuals used in fault detection are generated by comparing the actual measurements of the plant with the corresponding estimated quantities which are obtained by estimation. The requirement of robust residual generation naturally leads to the problem of robust estimation which is the key factor in guaranteeing robust fault detection.

Different estimator-based residual generation approaches can be used and the widely used methods include the parity space approach [?, ?, ?, ?], the fault detection filter approach [?, ?, ?], and the unknown input observer approach [?, ?, ?, ?]. The motivation behind each of these approaches is to distinguish the effects of faults from those caused by plant uncertainty and external disturbances and thus to achieve robust fault detection. Each of these approaches models the uncertainties as extra disturbance input terms and seeks either to completely decouple the effects of uncertainties and disturbance inputs from those caused by faults or to minimize certain norms of the transfer function matrix from disturbance inputs to the residual signals. A necessary condition for the complete decoupling of disturbances from the residual signals is that the number of disturbances not exceed the number of measurements which limits the application scope of the fault detection techniques based on complete decoupling. Another approach [?, ?] is to model the uncertainties as complex uncertainties with bounded magnitude and to solve the robust fault detection problem by using the small gain theorem. As discussed below, this approach may lead to conservative results. In this paper, the robust fault detection is performed by using a robust H_∞ estimation framework based on multiplier theory [?, ?, ?, ?, ?, ?] (essentially mixed structured singular value theory) which for real parametric uncertainties is less conservative than approaches based on the small gain theorem.

Two types of faults may occur in a given dynamic system. Hard failure, or abrupt fault, is easy to detect because the faulty element ceases functioning completely. Soft failure, or incipient fault, on the other hand is more difficult to detect. The robust fault detection framework presented in this paper is expected to be able to capture small incipient faults.

The goal of estimation is to reconstruct certain variables of a dynamic system using the available, noise corrupted measurements. The estimators can be designed with different performance criteria to satisfy the specific application requirements. The well-known Kalman filter is an estimator that minimizes the covariance of the estimation error by assuming a white noise disturbance model with a fixed covariance and is hence effective in rejecting wide-band disturbances. H_∞ estimation assumes a deterministic disturbance model consisting of bounded energy ℓ_2 signals and should be used if the disturbances present are mainly narrow-band disturbances.

No matter what performance criterion is used, the estimator design is based on a design model which cannot be obtained exactly, and hence, the performance of the estimator may be undermined when it is applied to the real system. As a result, the model uncertainty, both parametric and complex (i.e., unmodelled dynamics), needs to be accounted for explicitly. Some research has explicitly sought to take into account model uncertainty in the design of estimators. In Refs. xie:1991 and xie:1994, fixed quadratic Lyapunov functions are used to design respectively, robust H_∞ estimators and robust H_2 estimators (i.e., robust Kalman filters) for systems with parametric uncertainty. While Ref. xie:1991 focuses on linear, continuous-time systems, Ref. xie:1994 considers linear, discrete-time systems. Ref. the:1996 designs robust H_2 estimators for linear discrete-time, time-varying systems with parametric uncertainties. In Ref. col:1998, robust H_2 estimation is studied based on multiplier theory.

It is now well known that robust design of fixed quadratic Lyapunov functions is intimately related to the small gain theorem and assumes that the uncertainty is arbitrarily time-varying or complex. This assumption can lead to very conservative designs for systems with time-invariant, parametric uncertainty. However, mixed structured singular value theory and the associated multiplier theory [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**] are based on parameter-dependent Lyapunov functions and can lead to much less conservative results [**?**, **?**].

In this paper, the problem of robust H_∞ estimation will be studied by using multiplier theory. In particular, by using the Popov-Tsypkin multiplier [**?**, **?**, **?**], the robust H_∞ problem is formulated as a Riccati equation feasibility problem in which a cost function is minimized subject to a Riccati equation constraint. A continuation algorithm [**?**] that uses quasi-Newton (BFGS) corrections [**?**] is developed to solve the minimization problem. The algorithm is initialized with an H_∞ estimator obtained by solving a single algebraic Riccati equation [**?**] corresponding to the nominal system. The initializing multiplier matrices are obtained by solving a linear matrix inequality (LMI) [**?**].

The paper begins in Section 2 with the formulation of the robust H_∞ estimation problem for uncertain, linear discrete-time systems. Section 3 formulates the robust H_∞ estimation problem as a Riccati equation feasibility problem based on multiplier theory and a continuation algorithm is developed to solve the problem. Section 4 discusses the application of robust H_∞ estimation to robust fault detection of dynamic systems with uncertainties and external disturbance inputs. Section 5 presents a practical application and Section 6 concludes the paper.

Notation and Definitions

$\mathcal{R}, \mathcal{C}, \mathcal{Z}^+$	real numbers, complex numbers, nonnegative integers
$\mathcal{R}^{m\times n}, \mathcal{C}^{m\times n}$	$m \times n$ real matrices, complex matrices
$\mathcal{D}^n, \mathcal{S}^n$	$n \times n$ real diagonal, symmetric matrices
$\mathcal{N}^n, \mathcal{P}^n$	$n \times n$ nonnegative definite, positive definite matrices
$0,\ I$	zero matrix, identity matrix
tr	trace
$M_2 > M_1$	$M_2 - M_1$ positive definite
$M_2 \geq M_1$	$M_2 - M_1$ nonnegative definite
$\dim(M)$	dimension of M
$\|G(z)\|_\infty$	$\sup_{\theta\in[0,2\pi]} \sigma_{max}(G(e^{j\theta}))$
$\mathrm{Vec}(\cdot)$	the standard column stacking operator for matrices
$z_{i,j}$	(i,j) element of matrix Z

2. The Robust H_∞ Estimation Problem

Consider the discrete-time linear uncertain system

$$x_p(k+1) = (A_p + \Delta A_p)x_p(k) + D_{p,1}w(k), \quad k \in \mathcal{Z}^+ \tag{1}$$

$$y_p(k) = (C_p + \Delta C_p)x_p(k) + D_{p,2}w(k) \tag{2}$$

where $x_p \in \mathcal{R}^{n_p}$ is the state vector, $y_p \in \mathcal{R}^{p_p}$ denotes the plant measurements, $w \in \mathcal{R}^d$ denotes an ℓ_2 disturbance signal, and the uncertainties ΔA_p and ΔC_p are as defined in Appendix A. It is desired to design a predictive filter of the form

$$x_e(k+1|k) = A_e x_e(k|k-1) + W y_p(k) \tag{3}$$

to estimate (in some sense to be defined) the state vector x_p, where $A_e \in \mathcal{R}^{n_p\times n_p}$ and $W \in \mathcal{R}^{n_p\times p_p}$ are the filter parameters to be determined.

Define the estimation error as

$$e(k) \triangleq x_p(k) - x_e(k|k-1), \tag{4}$$

which using (??), (??), and (??) can be shown to obey the evolution equation

$$e(k+1) = A_e e(k) + (A_p + \Delta A_p - W\Delta C_p - A_e - WC_p)x_p(k) + (D_{p,1} - WD_{p,2})w(k). \quad (5)$$

Next, define the error output $z \in \mathcal{R}^q$ as $z(k) \triangleq E_p e(k)$. Then augmenting (??) with (??) yields

$$x(k+1) = (A + \Delta A)x(k) + Dw(k), \quad (6)$$

$$z(k) = Ex(k), \quad (7)$$

where

$$x(k) = \begin{bmatrix} x_p(k) \\ e(k) \end{bmatrix}, \quad A = \begin{bmatrix} A_p & 0 \\ A_p - A_e - WC_p & A_e \end{bmatrix}, \quad (8)$$

$$D = \begin{bmatrix} D_{p,1} \\ D_{p,1} - WD_{p,2} \end{bmatrix}, \qquad E = \begin{bmatrix} 0 & E_p \end{bmatrix}, \quad (9)$$

and ΔA is defined in Appendix A.

The Robust H_∞ Estimation Problem. Find A_e and W such that

1. the augmented system (??) is asymptotically stable over the uncertainty set $\mathcal{U}_A$; and
2. the H_∞ performance satisfies

$$\sup_{\Delta A \in \mathcal{U}_A} \|G_{zw}(z)\|_\infty < \frac{1}{\gamma}, \quad (10)$$

   where $\gamma > 0$, and $G_{zw}(z) \in \mathcal{C}^{q\times d}$ is the transfer function matrix from the disturbance signal $w(\cdot)$ to the performance variable $z(\cdot)$.

3. Robust H_∞ Estimation

One of the most important developments in robustness analysis was the development of mixed structured singular value concepts which are based on multiplier theory [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?]. These concepts were developed for both real parametric and unstructured uncertainty (e.g., unmodelled dynamics) and greatly reduce the conservatism of previous techniques used for robust analysis and synthesis. This section begins by restating a Riccati equation robust stability condition for uncertain linear, discrete-time systems based on the Popov-Tsypkin multiplier. These results are then used to formulate the robust H_∞ problem as a Riccati equation feasibility problem which is solved by a continuation algorithm that uses quasi-Newton (BFGS) corrections.

3.1. Riccati Equation Robust Stability Conditions

Consider the standard uncertainty feedback configuration given in Fig. 1, where $G(z) \in \mathcal{C}^{m\times m}$ is square, asymptotically stable and $G(z) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$. Assume that the uncertainty Δ satisfies

$$\Delta \in \mathcal{U} \triangleq \{\Delta \in \mathcal{R}^{m\times m} : M_1 \leq \Delta \leq M_2\}, \tag{11}$$

where $M_1, M_2 \in \mathcal{D}^m$ and $M \triangleq M_2 - M_1$ is nonnegative definite.

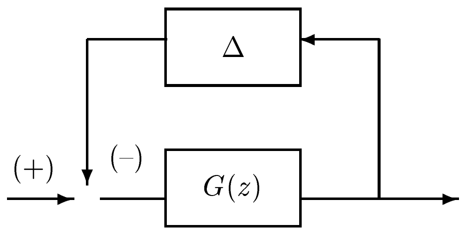

Figure 1: Standard Uncertainty Feedback Configuration

The next theorem provides a Riccati equation robust stability condition for the uncertain feedback interconnection of $G(z)$ and Δ in terms of the Popov-Tsypkin multiplier [?, ?, ?] which can be written in the transfer function matrix form as

$$M(z) = H + N\frac{z-1}{z}, \tag{12}$$

where $H \in \mathcal{D}^m$, $N \in \mathcal{D}^m$ with $H > 0$ and $N \geq 0$.

Theorem 1. [?] Assume $G(z)$ is asymptotically stable. If there exist real diagonal $H > 0$, $N \geq 0$, $P > 0$ and $\epsilon > 0$ such that $D_{\rm a}(H,N) + D_{\rm a}(H,N)^T - B_{\rm a}^T P B_{\rm a} > 0$ and

$$\begin{aligned} P \;=\; & A_{\rm a}^T P A_{\rm a} + [B_{\rm a}^T P A_{\rm a} - C_{\rm a}(H,N)]^T \\ & [D_{\rm a}(H,N) + D_{\rm a}(H,N)^T - B_{\rm a}^T P B_{\rm a}]^{-1}[B_{\rm a}^T P A_{\rm a} - C_{\rm a}(H,N)] + \epsilon I, \end{aligned} \tag{13}$$

where

$$A_a = \begin{bmatrix} A - BM_1C & 0 \\ (M_2 - M_1)C & 0 \end{bmatrix}, \qquad B_a = \begin{bmatrix} B \\ I \end{bmatrix}, \tag{14}$$

$$C_a(H,N) = \begin{bmatrix} (H+N)(M_2 - M_1)C & N \end{bmatrix}, \qquad D_a(H,N) = H + N, \tag{15}$$

then the negative feedback interconnection of $G(z)$ and Δ is asymptotically stable for all $\Delta \in \mathcal{U}$.

Remark 1. Although we have explicitly only considered real parametric uncertainty in Theorem **??**, this theorem is easily modified to the more general case of mixed (i.e., real parametric and unstructured) uncertainty. This modification requires that the elements of N corresponding to the unstructured uncertainty be zero.

3.2. Robust H_∞ Estimation

The uncertain system given by equations (**??**) and (**??**) can be represented in the form of Fig. 2 with

$$G(z) \sim \left[\begin{array}{c|cc} A & B_0 & D \\ \hline C_0 & 0 & 0 \\ E & 0 & 0 \end{array}\right] \triangleq \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right]. \tag{16}$$

Note that matrices A, D, E, B_0, and C_0 are as defined in equations (**??**), (**??**) and (**??**).

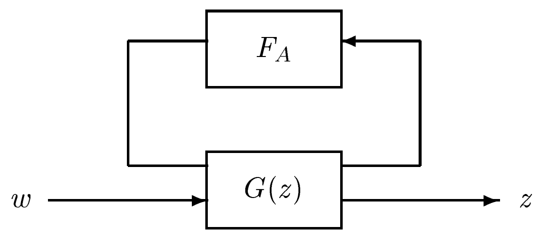

Figure 2: Representation of the Uncertain Filter System

To consider H_∞ performance, a fictitious complex uncertainty block Δ_p is inserted into Fig. 2 [**?**, **?**] as shown in Fig. 3. It is assumed that $\sigma_{max}(\Delta_p) < \gamma$. For ease of presentation, assume that $\dim(z) = \dim(w) = q$, such that $\Delta_p \in \mathcal{C}^{q\times q}$. Define

$$\tilde{M}_1 \triangleq \text{block-diag}\{M_1, -\gamma I_q\}, \quad \tilde{M}_2 \triangleq \text{block-diag}\{M_2, -\gamma I_q\}. \tag{17}$$

In order to account for the performance block Δ_p, the multiplier matrices H and N are redefined

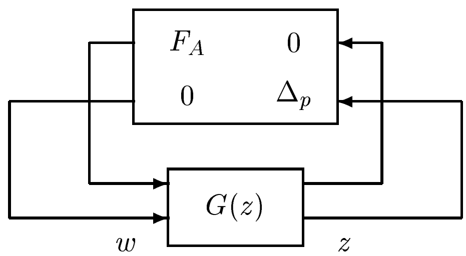

Figure 3: Uncertain Filter System with H_∞ 'Performance Block'

as

$$H \triangleq \text{block-diag}\{H_1,\ H_2\}, \tag{18}$$

$$N \triangleq \text{block-diag}\{N_1,\ 0_q\} \tag{19}$$

where $H_1 \in \mathcal{D}^{(r+s)\times(r+s)}$, $H_2 \in \mathcal{R}^{q\times q}$ satisfies $H_2\Delta_p = \Delta_p H_2$, and $N_1 \in \mathcal{D}^{(r+s)\times(r+s)}$. The next theorem forms the theoretical basis for robust H_∞ estimator synthesis.

Theorem 2. Assume $G(z)$ is asymptotically stable. If there exist real diagonal $H > 0$ and $N \geq 0$, $P > 0$ and $\epsilon > 0$ such that $D_{\rm a}(H,N) + D_{\rm a}(H,N)^T - B_{\rm a}^T P B_{\rm a} > 0$ and

$$\begin{aligned} P \ = \ & A_{\rm a}^T P A_{\rm a} + [B_{\rm a}^T P A_{\rm a} - C_{\rm a}(H,N)]^T \\ & \cdot (D_{\rm a}(H,N) + D_{\rm a}(H,N)^T - B_{\rm a}^T P B_{\rm a})^{-1} [B_{\rm a}^T P A_{\rm a} - C_{\rm a}(H,N)] + \epsilon I, \end{aligned} \tag{20}$$

where

$$A_a = \begin{bmatrix} \tilde{A} - \tilde{B}\tilde{M}_1\tilde{C} & 0 \\ (\tilde{M}_2 - \tilde{M}_1)\tilde{C} & 0 \end{bmatrix}, \qquad B_a = \begin{bmatrix} \tilde{B} \\ I \end{bmatrix}, \tag{21}$$

$$C_a(H,N) = \begin{bmatrix} (H+N)(\tilde{M}_2 - \tilde{M}_1)\tilde{C} & N \end{bmatrix}, \qquad D_a(H,N) = H + N, \tag{22}$$

then the uncertain system of Fig. 3 is asymptotically stable for each $\Delta A \in \mathcal{U}_A$. In addition,

$$\sup_{\Delta A \in \mathcal{U}_A} \|G_{zw}(z)\|_\infty < \frac{1}{\gamma}. \tag{23}$$

Proof. Follows from Theorem 1 and the main loop theorem [**?**, **?**]. □

Theorem 2 poses the robust H_∞ estimation problem as a Riccati Equation Feasibility Problem (REFP) []. As discussed in Refs. col:1997b and col:1997c, an approach to solving the REFP can be based on solving an optimization problem

$$\min_{\Delta A \in \mathcal{U}_A} \ \mathcal{J}(A_e, W) \ \text{ subject to Equation (??)}. \tag{24}$$

Here, the robust H_∞ performance $\mathcal{J}(\cdot)$ can be chosen as the artificial cost function [**?**, **?**]

$$\mathcal{J}(A_e, W) \triangleq \text{tr} P. \tag{25}$$

Auxiliary minimization problem. The robust H_∞ estimation problem can be cast as an auxiliary minimization problem in which (**??**) is minimized subject to the constraint represented by equation (**??**).

To characterize the extremals define the Lagrangian

$$\mathcal{L}(\epsilon, A_e, W, H, N, P, Q) = \mathcal{J}(A_e, W) + \text{tr} Q[-P + \text{the R.H.S. of (??)}]. \tag{26}$$

Thus the necessary conditions for a solution to (??) are given by

$$\frac{\partial \mathcal{L}}{\partial \epsilon}=0, \quad \frac{\partial \mathcal{L}}{\partial W}=0, \quad \frac{\partial \mathcal{L}}{\partial A_e}=0, \quad \frac{\partial \mathcal{L}}{\partial H}=0, \quad \frac{\partial \mathcal{L}}{\partial N}=0, \quad \frac{\partial \mathcal{L}}{\partial P}=0, \quad \frac{\partial \mathcal{L}}{\partial Q}=0. \tag{27}$$

3.3. A Continuation Algorithm

To solve the above auxiliary minimization problem, a continuation algorithm is developed. The correction steps of the algorithm are performed by using the BFGS inverse Hessian update which requires only gradient information. The line search algorithm includes a constraint checking routine which guarantees that the cost function remains defined at every point in the line search process.

It should be noted that in equation (??), $\frac{\partial \mathcal{L}}{\partial Q}=0$ recovers Riccati equation (??) and $\frac{\partial \mathcal{L}}{\partial P}=0$ results in a Lyapunov equation in Q whose coefficients and forcing matrices are functions of P. In the continuation algorithm, the Riccati equation and Lypunov equation are solved by using the discrete-time Riccati and Lyapunov equation solvers respectively. The detailed continuation algorthm used to solve this robust H_∞ estimation problem is presented in Appendix B.

4. Robust Fault Detection

In this section, the robust H_∞ estimation framework presented in the previous sections will be applied to the robust fault detection for uncertain dynamic systems with disturbance inputs. Consider the uncertain discrete-time system

$$x_p(k+1) = (A_p+\Delta A_p)x_p(k)+D_{p,1}w(k)+R_{p,1}f(k), \tag{28}$$

$$y_p(k) = C_p x_p(k)+D_{p,2}w(k)+R_{p,2}f(k) \tag{29}$$

where x_p, y_p, and w are as discussed in previous sections, and $f \in \mathcal{R}^{n_f}$ is the fault vector. The term $R_{p,1}f(k)$ represents actuator and component faults while $R_{p,2}f(k)$ denotes the sensor faults. The fault distribution matrices $R_{p,1}$ and $R_{p,2}$ are assumed to be known.

It should be mentioned that the determination of the fault vector f and the fault distribution matrices $R_{p,1}$, $R_{p,2}$ are normally determined on a case-by-case basis by inspection of the state-space model and the characteristics of the particular process. A more general design procedure can be performed by employing *component fault analysis* techniques [?] which guarantee that a complete set of fault effects is used.

The robust fault detection problem is to generate a robust residual signal $r(k)$ that satisfies

$$\|r(k)\|_p \leq J_{th} \quad \text{if} \quad f(k)=0, \tag{30}$$

$$\|r(k)\|_p \; > \; J_{th} \quad \text{if} \quad f(k) \neq 0, \tag{31}$$

where $\|\cdot\|_p$ denotes the p norm of a Lebesgue signal and J_{th} is a threshold value. The residual generated is given by the following equation if estimator (**??**) is applied to the system described by (**??**) and (**??**).

$$\begin{aligned} r(k) \; &= \; y_p(k) - C_p x_e(k) \\ &= \; C_p e(k) + D_{p,2} w(k) + R_{p,2} f(k), \end{aligned} \tag{32}$$

where the estimation error $e(k)$ is governed by

$$\begin{aligned} e(k+1) \; &= \; (A_{e,k} - WC_p)e(k) + (\Delta A_p + A_p - A_{e,k})x_p(k) \\ &+ \; (D_{p,1} - WD_{p,2})w(k) + (R_{p,1} - WR_{p,2})f(k). \end{aligned} \tag{33}$$

It is clear from equations (**??**)-(**??**) and (**??**) that if E_p is chosen as C_p, then

$$r(k) = z(k) + D_{p,2} w(k) + R_{p,2} f(k), \tag{34}$$

which satisfies the following norm inequality condition

$$\|r\|_{2,[N_0,N]} \leq \|z\|_{2,[N_0,N]} + \|D_{p,2} w\|_{2,[N_0,N]} + \|R_{p,2} f\|_{2,[N_0,N]}, \tag{35}$$

where $[N_0, N]$ corresponds to a certain time interval. Note that similar norm conditions used for fault detection can be found in Refs. fra:1994b and ran:1998. If $f(k) = 0$, equation (**??**) reduces to

$$\|r\|_{2,[N_0,N]} \leq \|z\|_{2,[N_0,N]} + \|D_{p,2} w\|_{2,[N_0,N]}. \tag{36}$$

Note that

$$\|z\|_{2,[N_0,N]} \leq \sup_{\Delta A \in \mathcal{U}_A} \|G_{zw}\|_\infty \|w\|_{2,[N_0,N]} < \frac{1}{\gamma} \|w\|_{2,[N_0,N]}, \tag{37}$$

and

$$\|D_{p,2} w\|_{2,[N_0,N]} \leq \sigma \text{max}(D_{p,2}) \|w\|_{2,[N_0,N]}. \tag{38}$$

Thus, the threshold can be chosen as

$$J_{th} \triangleq (\frac{1}{\gamma} + \sigma_{max}(D_{p,2})) \|w\|_{2,[N_0,N]}. \tag{39}$$

Robust fault detection can be accomplished by comparing $\|r\|_{2,[N_0,N]}$ with J_{th}. A fault occurs if $\|r\|_{2,[N_0,N]} > J_{th}$, i.e.,

$$\|r\|_{2,[N_0,N]} > J_{th}, \quad \text{fault occurred.} \tag{40}$$

Remark 2. It should be noted that the threshold value defined in equation (**??**) depends on the norm of noise signal w. In practice, it is more feasible to use an upper bound on $\|w\|_{2,[N_0,N]}$ to calculate the threshold value and in which case J_{th} collapses to a constant threshold. The application in Section 5 uses an upper bound of $\|w\|_{2,[N_0,N]}$ in calculating the threshold. In addition, It should be recognized that the fault detection condition given by equation (**??**) is only a sufficient condition. It is possible that small faults occur in a system but the residual signal does not surpass the threshold which results in missed detection.

5. Illustrative Example

A practical robust fault detection application is presented in this section to illustrate the robust H_∞ estimator design using the Popov-Tsypkin multiplier and the application of the robust H_∞ estimator to robust fault detection of dynamic systems. The results are generated with the algorithm given in Appendix B. The application considered here is a linearized discrete-time model of a simplied longitudinal flight control system [**?**]. This flight control system has three state variables and can be represented by the following state space equations

$$x_p(k+1) = (A_p + \Delta A_p)x_p(k) + D_{p,1}w(k), \quad k \in \mathcal{Z}^+ \tag{41}$$

$$y_p(k) = C_p x_p(k) + D_{p,2}w(k) \tag{42}$$

where the elements of the state variable vector $x(k) \triangleq \begin{bmatrix} \eta_y & \omega_z & \delta_z \end{bmatrix}^T$ are normal velocity η_y, pitch rate ω_z, and pitch angle δ_z. Each of the three state variables is measured by a sensor and these measurements are used as feedback signals. The performance of this flight control system depends on the sensors. In this application, the state of the sensors are monitored through an estimator and any malfunction can be captured by the robust fault detection framework presented.

In the remainder of this section, a robust H_∞ estimator is designed for this flight system. By comparing the measured output with the estimated output, a residual signal is generated which is compared against a threshold value. If the residual surpasses the threshold value, then a fault can be declared.

The system parameter matrices are

$$A_p = \begin{bmatrix} 0.8950 & -0.1083 & -0.3872 \\ 0.0015 & 0.8912 & -0.0672 \\ 0 & 0.7368 & 0 \end{bmatrix}, \qquad C_p = I_{3\times 3},$$

$$D_{p,1} = \text{diag}\{0.1,\ 0.1,\ 0.01\}, \qquad D_{p,2} = 0.1 \times I_{3\times 3}.$$

The uncertainty matrix $\Delta A_p = -B_{A_p} F_{A_p} C_{A_p}$, where

$$
\begin{aligned}
B_{A_p} &= -\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad C_{A_p} = \begin{bmatrix} I_{3\times 3} \\ I_{3\times 3} \end{bmatrix}, \\
F_{A_p} &= \mathrm{diag}\{\delta_1,\ \delta_2,\ \delta_3,\ \delta_4,\ \delta_5,\ \delta_6\}
\end{aligned}
$$

with $-0.06961 \le \delta_1 \le 0.06961$, $-0.00868 \le \delta_2 \le 0.00868$, $-0.03011 \le \delta_3 \le 0.03011$, $-0.00012 \le \delta_4 \le 0.00012$, $-0.06931 \le \delta_5 \le 0.06931$, and $-0.00523 \le \delta_6 \le 0.00523$. Note that the uncertain parameters δ_1 through δ_6 correspond to $\pm 8\%$ parameter fluctuations in the first two rows of matrix A_p.

It is desired to design a predictive filter of the form

$$
x_e(k+1|k) = A_e x_e(k|k-1) + W y_p(k) \tag{43}
$$

for which the estimation error is given by (**??**). For this particular example, the error output is defined as $z(k) = E_p e(k)$ where $E_p \triangleq I_{3\times 3} = C_p$ which satisfies the residual equation (**??**). The robust H_∞ estimation problem is to design A_e and W in (**??**) such that the uncertain system (**??**) is asymptotically stable for each ΔA in the uncertainty set and the artificial cost (**??**) is minimized.

As discussed in the previous section, the algorithm is intialized with an H_∞ estimator obtained by solving a single Riccati equation. The intializing estimator is given by

$$
x_e(k+1|k) = A_p x_e(k|k-1) + W(y_p(k) - C_p x_e(k)) \tag{44}
$$

where

$$
W = \begin{bmatrix} 0.4475 & -0.0541 & -0.0038 \\ 0.0008 & 0.4456 & -0.0007 \\ 0.0000 & 0.3684 & 0.0000 \end{bmatrix}. \tag{45}
$$

After a LMI test of the initializing estimator, the feasible initial values of the multiplier matrices are obtained as $H = 35.1746 \times I_{12\times 12}$ and $N = 1.5474 \times \mathrm{diag}\{I_{9\times 9},\ 0_{3\times 3}\}$.

The robust H_∞ estimator generated by using the algorithm are given by

$$
W = \begin{bmatrix} 0.7167 & -0.1021 & -0.1594 \\ -0.0035 & 0.6669 & 0.0220 \\ -0.0164 & 0.5332 & 0.0489 \end{bmatrix}, \qquad A_e = \begin{bmatrix} 0.1917 & 0.0185 & -0.1444 \\ 0.0005 & 0.1872 & -0.0269 \\ 0.0117 & 0.1531 & -0.0092 \end{bmatrix}. \tag{46}
$$

Figures 4, 5, and 6 show the comparison of the estimation errors of velocity, pitch rate, and pitch angle respectively by using the nominal H_∞ estimator and the robust H_∞ estimator. It can be seen from these figures that the robust H_∞ estimator has better performance than the nominal H_∞ estimator. In producing these estimation results, the uncertain parameters $\{\delta_1,\ \delta_2,\ \delta_3,\ \delta_4,\ \delta_5,\ \delta_6\}$