# Synthesis of Fixed-Architecture, Robust $H_2$ and $H_\infty$ Controllers

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E. G. Collins, Jr., and D. Sadhukhan, "Synthesis of Fixed-Architecture, Robust $H_2$ and $H_\infty$ Controllers," *Mathematical Problems in Engineering*, Vol. 6, pp. 125-144, 2000.

# Synthesis of Fixed-Architecture, Robust $H_2$ and $H_\infty$ Controllers

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**Abstract**

This paper discusses the synthesis of fixed-architecture controllers that guarantee either robust $H_2$ or $H_\infty$ performance. The synthesis is accomplished by solving a Riccati equation feasibility problem resulting from mixed structured singular value theory with Popov mulipliers. It is seen that the numerical algorithm for robust $H_\infty$ performance is much more complex that that for robust $H_2$ performance.

## 1 Introduction

This paper considers the design of robust controllers using the state space Popov analysis criterion which is based on the Popov stability multiplier $W(s) = H^2 + Ns$. This is a special case of mixed structured singular value synthesis [7, 9]. Both robust $H_2$ and $H_\infty$ performance are considered. $H_\infty$ performance has been considered by introducing a fictitious uncertainty block [5, 16] into the standard uncertainty feedback configuration shown in Fig. 1. Both formulations require the minimization of a cost functional subject to a Riccati equation constraint [3, 4]. This formulation has several advantages. First, compensator order and architecture can be specified a priori. In addition, both the controller and multiplier parameters can be optimized simulatenously which avoids $M$-$K$ (i.e., multiplier-controller) iteration.

For robust $H_2$ performance the cost function that is minimized is an upper bound on the $H_2$ performance over the uncertainty set. For $H_\infty$ performance, an artificial cost function is used since the $H_\infty$ performance enters into the problem through the introduction of a fictitious uncertainty block. Though the formulations for the robust $H_2$ and $H_\infty$ problems are very similar, the gradient and consequently the Hessian expressions for the $H_\infty$ formulation are much more complex resulting in more computationally intensive algorithms.

Because of positive definite constraints on the Riccati equation solution, standard descent techniques cannot be used to solve the resulting optimization problem. Hence, probability-one homotopy algorithms have been formulated [3, 4]. These algorithms have desirable properties when applied to controller design. First, they can be initialized with any feasible multiplier and stabilizing controller. Also, each controller computed as the homotopy curve is traversed is physically meaningful. In particular, for the robust $H_2$ performance each controller along the homotopy path guarantees a specified degree of robust stability while for the robust $H_\infty$ performance problem each controller guarantees a specified degree of both robust stability and robust performance.

The paper is organized as follows. Section 2 presents the general form of a Riccati Equation Feasibility Problem (REFP) for the synthesis of robust controllers using the Popov multiplier and provides the specific formulations for the synthesis of controllers for robust $H_2$ and $H_\infty$ performance. Section 3 describes the solution approach via probability-one homotopy algorithms. Section 4 shows the result of both robust $H_2$ and $H_\infty$ control design for a standard benchmark problem. Conclusions are presented in Section 5.

### Notation and Definitions

| | |
|---|---|
| $\mathcal{R}$ | real numbers |
| $\mathcal{R}^{r\times s}$ | $r\times s$ real matrices |
| $\mathcal{R}^{r}$ | $\mathcal{R}^{r\times 1}$ |
| $\mathcal{C}$ | complex numbers |
| $\mathcal{C}^{r\times s}$ | $r\times s$ complex matrices |
| $\mathcal{C}^{r}$ | $\mathcal{C}^{r\times 1}$ |
| $\mathcal{D}^{r\times r}$ | $r\times r$ diagonal matrices |
| tr | trace |
| $0_{r\times s}$ | $r\times s$ zero matrix |
| $()^T$ | transpose |
| $()^*$ | complex conjugate transpose |
| $I_r$ | $r\times r$ identity |
| $Z_2 > Z1$ | $Z_2 - Z_1$ positive definite |
| $Z_2 \geq Z_1$ | $Z_2 - Z_1$ nonnegative definite |

## 2 Riccati Equation Characterization of the Popov Criterion

Consider the linear uncertain system of the form

$$\dot{x} = (A+\Delta A)x + Dw \quad (1)$$
$$z = Ex \quad (2)$$

where $x \in \mathcal{R}^n$, $A \in \mathcal{R}^{n\times n}$ denotes the nominal dynamics matrix, $\Delta A$ denotes the parametric uncertainty belonging to a specified set $\mathcal{U}$, $w(t)$ is an unit intensity white noise signal, and $z(t) \in \mathcal{R}^q$ is a vector of outputs. In this case, the uncertainty set $\mathcal{U}$ is defined by

$$\mathcal{U} \triangleq \{\Delta A \in \mathcal{R}^{n\times n} : \Delta A = -BFC, M_1 < F < M_2\} \quad (3)$$

The matrices $B \in \mathcal{R}^{n\times m}$ and $C \in \mathcal{R}^{m\times n}$ denote the structure of uncertainty, and $F \in \mathcal{D}^{m\times m}$ is an uncertain diagonal matrix bounded by the diagonal matrices $M_1$ and $M_2$. In the following formulation, the diagonal values of $M_1$ and $M_2$ represent the guaranteed limits of robust stability in the $H_2$ performance case and guaranteed limits of both robust stability and performance in the $H_\infty$ performance case. In the following a stability test based on the Popov multiplier $W(s) = H^2 + Ns$, is developed.

**Theorem 1** *[7] Let $M_1, M_2, N, H \in \mathcal{D}^m$ be such that $M_2 - M_1$ is positive definite, $N$ is nonnegative definite and $\epsilon$ is a positive scalar. If there exists a nonnegative definite matrix $P$ satisfying*

$$\begin{aligned} 0 = \; & (A - BM_1C)^T P + P(A - BM_1C) + \\ & [B^TP - H^2C - NC(A - BM_1C)]^T \cdot Y^{-1} \cdot \\ & [B^TP - H^2C - NC(A - BM_1C)] + \epsilon R. \end{aligned} \quad (4)$$

*where*

$$R \triangleq E^TE \quad (5)$$

*and*

$$Y = [2H^2(M_2 - M_1)^{-1} + NCB + B^TC^TN] > 0 \quad (6)$$

*then the uncertain system described by (1)-(2) is asymptotically stable for all $\Delta A \in \mathcal{U}$. In this case the worst-case overbound of the $H_2$ cost function is given by*

$$J(\mathcal{U}) \leq \bar{J} \triangleq \frac{1}{\epsilon}\mathrm{tr}[P + C^T(M_2 - M_1)NC)]V \quad (7)$$

*where*

$$V \triangleq DD^T \quad (8)$$

Hence the Popov analysis criterion has been posed as a Riccati Equation Feasibility Problem (REFP).

## 3 Dynamic Output Feedback Controllers

Consider the $n_x$th-order stabilizable and detectable uncertain plant

$$\dot{x}_p = (A_p + \Delta A_p)x_p + B_pu + D_1w \quad (9)$$
$$y = C_px_p + D_2w \quad (10)$$
$$z_p = E_1x_p \quad (11)$$

where $u \in \mathcal{R}^{n_u}$, $w \in \mathcal{R}^{n_w}$, $y \in \mathcal{R}^{n_y}$ is the output variable and $z_p \in \mathcal{R}^{n_z}$ is the performance variable. The synthesis problem is to determine an $n_c$th-order dynamic compensator

$$\dot{x}_c = A_cx_c + B_cy \quad (12)$$
$$u = -C_cx_c \quad (13)$$

such that the closed loop system in Eqs. (9-13) is asymptotically stable for all $\Delta A_p = B_0FC_0 \in \mathcal{U}$, where $B_0$ and $C_0$ represent the structure of the uncertainty.

**Formulation for $H_2$ Performance**
The closed loop system can be represented by

$$\dot{x} = (A+\Delta A)x + Dw \quad (14)$$
$$z = Ex \quad (15)$$

where assuming negative feedback

$$x \triangleq \begin{bmatrix} x_p \\ x_c \end{bmatrix}, \quad A = \begin{bmatrix} A_p & -B_pC_c \\ B_cC_p & A_c \end{bmatrix} \quad (16)$$

$$\Delta A \triangleq \begin{bmatrix} \Delta A_p & 0_{n\times n_c} \\ 0_{n_c\times n} & 0_{n_c\times n_c} \end{bmatrix} = BFC$$

where

$$B = \begin{bmatrix} B_0 \\ 0_{n_c\times m} \end{bmatrix} \; C = \begin{bmatrix} C_0 & 0_{m\times n_c} \end{bmatrix}. \quad (17)$$

Hence $G(s)$ in the standard uncertainty feedback configuration shown in Fig. 1 is given by

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \quad (18)$$

The $H_2$ cost function is defined with respect to the transfer function between the disturbance vector $w$ and the performance vector $z$. The closed loop $R$ and $V$ matrices are given by (5) and (8) where

$$E = \left[ \begin{array}{cc} E_1 & -E_2C_c \end{array} \right], \quad D = \left[ \begin{array}{c} D_1 \\ B_cD_2 \end{array} \right] \quad (19)$$

and $E_2^TE_2$ is the input weighting matrix.

**Formulation for $H_\infty$ Performance**

The above problem may be reformulated to design for $H_\infty$ performance. The closed-loop system in Fig. 2(a) may be represented by the following equations.

$$\dot{x} = Ax - Bu_1 + Dw \quad (20)$$

where $A$ is given by (16), $B$ by (17) and $D$ by (19).

$$u_1 = \Delta_u y_1, \quad y_1 = Cx \quad (21)$$

where $\Delta_u$ represents the uncertainty and $C$ is given by (17).

$$z = z_p = \tilde{E}x \quad (22)$$

where

$$\tilde{E} = \left[ \begin{array}{cc} E_1 & 0_{n_z \times n_c} \end{array} \right]. \quad (23)$$

It is generally desired to keep the performance variable $z$ in (22) small while keeping the the control effort $u$ in (13) 'reasonable'. Hence the closed loop system in Fig. 2(a) is rearranged as shown in Fig. 2(b).

The output vector $\tilde{y}$ may now be written as

$$\tilde{y} = \left[ \begin{array}{c} y_1 \\ z \\ u \end{array} \right] = \left[ \begin{array}{c} C \\ \tilde{E} \\ \tilde{F} \end{array} \right] x = \tilde{C}x \quad (24)$$

where $\tilde{E}$ is given by (23) and

$$\tilde{F} = \left[ \begin{array}{cc} 0_{n_z \times n_x} & -C_c \end{array} \right]. \quad (25)$$

The loop between the output vector $\tilde{y}$ is now closed through a performance block $\Delta_p$ as shown in Fig. 2(c). The uncertainty block $\Delta_u$ and the performance block $\Delta_p$ are then lumped together to form one 'fictitious' uncertainty block $\Delta$, as shown in Fig.2(c) [5]. The system may now be represented by the following equations.

$$\dot{x} = Ax - \tilde{B}\tilde{u} \quad (26)$$

where $\tilde{u}$ is the input vector given by

$$\tilde{u} = \left[ \begin{array}{c} u_1 \\ w \end{array} \right] \quad (27)$$

and $\tilde{B}$ is given by

$$\tilde{B} = \left[ \begin{array}{cc} B & -D \end{array} \right] \quad (28)$$

$$\tilde{u} = \Delta\tilde{y} \quad (29)$$

where

$$\Delta = \left[\begin{array}{c|c} \Delta_u & 0 \\ \hline 0 & \Delta_p \end{array}\right] \quad (30)$$

$$\tilde{y} = \tilde{C}x \quad (31)$$

Hence $G(s)$ in the uncertainty feedback configuration shown in Fig. 1 is given by

$$G(s) \sim \left[\begin{array}{c|c} A & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right] \quad (32)$$

Now if a set of multipliers and controllers exist which satisfy the conditions of Theorem 1 then the closed loop system is stable for all $\Delta A \in \mathcal{U}$ and the $H_\infty$ performance condition $\left\| \left[ \begin{array}{c} z \\ u \end{array} \right] /w \right\|_\infty < 1/\gamma$ is satisfied for all $\|\Delta_p\|_\infty < \gamma$.

## 4 Riccati Equation Feasibility problem as an Optimization Problem

Both the $H_2$ and the $H_\infty$ formulation have been posed as a Riccati Equation Feasibility Problem (REFP) which may in turn be converted to the following optimization problem.

$$\min_{\epsilon,\theta,P} J(\epsilon, \theta, P) \text{ subject to (4).} \quad (33)$$

In the case of $H_2$ performance, $J$ is the upper bound on the $H_2$ cost function given by (7), and $\theta$ consists of the the free elements of the multipliers and controllers and is given by

$$\theta = \left[\text{vec}(H)^T,\ \text{vec}(N)^T,\ \text{vec}(A_c)^T,\ \text{vec}(B_c)^T,\ \text{vec}(C_c)^T\right]^T \quad (34)$$

In the case of the $H_\infty$ formulation we minimize the artificial cost function

$$J = \text{tr}(P). \quad (35)$$

To characterize the extremals, we define the Lagrangian

$$\mathcal{L}(\epsilon, \theta, P, Q) = J(\epsilon, \theta, P) + \text{tr}QW(\epsilon, \theta, P), \quad (36)$$

where $J$ is given by equation (7) or (35) and $W(\epsilon, \theta, P)$ denotes the right hand side of equation (4). The necessary conditions for a solution to (33) are given by

$$0 = \frac{\partial \mathcal{L}}{\partial \theta}, \quad 0 = \frac{\partial \mathcal{L}}{\partial \epsilon} \quad (37)$$

$$0 = \frac{\partial \mathcal{L}}{\partial Q} = (Z_1)^T P + P(Z_1) + P(BY^{-1}B)P + (Z^T Y^{-1} Z + \epsilon I) = \text{Equation 4.} \quad (38)$$

$$0 = \frac{\partial \mathcal{L}}{\partial P} = (RR)Q + Q(RR)^T + \beta I \quad (39)$$

where

$$\begin{aligned} X &= A - BM_1C \\ Z &= H^2C + NCX \\ Y &= 2H^2(M_2 - M_1)^{-1} + NCB + B^TC^TN \\ Z_1 &= X - BY^{-1}Z \\ R &= Z_1 + BY^{-1}Z \end{aligned}$$

## 5 Probability-One Homotopy Algorithms for Robust Controller Synthesis

Consider a function $F : \mathcal{R}^N \times \mathcal{R} \to \mathcal{R}^N$ that is $\mathcal{C}^2$. Given $\gamma_f$, it is desired to find $x \in \mathcal{R}^N$ such that

$$0 = F(x, \gamma_f). \quad (40)$$

This is a standard zero finding problem. In the context of the robust controller synthesis

$$x = (\theta, \epsilon), \quad \gamma_f = [M_{1f}, M_{2f}] \quad (41)$$

where $M_{1f}$ and $M_{2f}$ are the uncertainty bounds given in (3). In the formulation for $H_\infty$ performance this would represent the bounds for the uncertainty as well as performance, and

$$F(x, \gamma) = \left( \frac{\partial \mathcal{L}}{\partial \theta}, \frac{\partial \mathcal{L}}{\partial \epsilon} \right), \quad (42)$$

Note that $0 = \frac{\partial \mathcal{L}}{\partial Q}$ and $0 = \frac{\partial \mathcal{L}}{\partial P}$ are implicitly satisfied by choosing $P$ as the (maximal) solution of the Riccati equation (38) (or (4)) and $Q$ as the solution of the Lyapunov equation (39).

Let $x_0 = [\theta_0, \epsilon_0]$ represent a feasible multiplier, a stablizing compensator and a positive $\epsilon$. Furthermore let $\gamma_0$ be chosen small enough such that there exists a positive-definite solution $P_0$ to (4). (It is assumed that $\gamma_0 < \gamma_f$ such that the robustness problem is not trivial.) It should be fairly easy to find an initial feasible multiplier and stabilizing compensator if $\gamma_0$ is chosen sufficiently small.

We let

$$\gamma(\lambda) = (1 - \lambda)\gamma_0 + \lambda \gamma_f \quad (43)$$

and define the probability-one homotopy map $\rho : [0, 1) \times \mathcal{R}^N \to \mathcal{R}^N$ by

$$\rho(\lambda, x) = \lambda F(x, \gamma(\lambda)) + (1 - \lambda)(x - x_0). \quad (44)$$

The above homotopy map ensures that the Jacobian of the map is full rank at all points on the zero curve (except possibly at the end point). In the context of the robustness problem, this homotopy map also has the desirable property that it can be initialized with *any* feasible multiplier. In addition, for any $\lambda \in [0, 1)$ the corresponding point on the zero curve $(x, \lambda)$ corresponds to a controller and multiplier that guarantees the level of robustness (robustness and performance in case of the $H_\infty$ formulation) corresponding to $\gamma(\lambda)$ since the Riccati equation (4) (or (38)) with the constraints (6) are satisfied with $\gamma = \gamma(\lambda)$. Hence, each point on the zero curve $(0 = \rho(\lambda, x), \ \lambda \in [0, 1))$, is physically meaningful even though $F(x, \gamma_f) \neq 0$ for $0 < \lambda < 1$.

### 5.1 Probability-One Homotopy Algorithm

Complete details of the numerical algorithm are in Reference [14]. A sketch follows.

1. Set $\lambda = 0, \ x = x_0$.
2. Evaluate the homotopy map $\rho$ and the Jacobian of the homotopy map $D\rho$.
3. Predict the next point $Z^{(0)}$ on the homotopy zero curve using, e.g., a Hermite cubic interpolant.
4. For $k = 0, 1, 2, ...$ until convergence do
$$Z^{(k+1)} = Z^{(k)} - [D\rho(Z^{(k)})]^\dagger \rho(Z^{(k)}),$$
where $[D\rho(Z)]^\dagger$ is the Moore-Penrose pseudoinverse of $D\rho(Z)$. Let $(x_1, \lambda_1) = \lim_{k \to \infty} Z^k$.
5. If $\lambda_1 < 1$, then set $x = x_1, \ \lambda = \lambda_1$, and go to to step (2).
6. If $\lambda_1 > 1$, compute the solution $x$ at $\lambda = 1$ using, e.g., inverse linear interpolation.[14]

## 6 Numerical Example

To illustrate robust control synthesis with the probability-one homotopy algorithm, we consider the two-mass/spring benchmark system shown in Figure 3 with uncertain stiffness $k$. A control force acts on the body 1 and the position of body 2 is measured, resulting in a noncolocated control problem. This benchmark problem is discussed in detail in Reference [15].

The open-loop plant (for $m_1 = m_2 = 1$) is given by (9)-(11) where $z_p = x_2$ is the performance variable, $y$ is a noise corrupted measurement of $x_2$,

$$x_p = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \ A_p(k) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & 0 & 0 \\ k & -k & 0 & 0 \end{bmatrix},$$

$$B_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$C_p = E_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

It is assumed that $k = k_{\text{nom}} + \Delta k$. The perturbation in $A_p(k)$ due to a change $\Delta k$ in the stiffness element $k$ from the nominal value $k_{\text{nom}}$ is given by

$$A_p(k) - A_p(k_{\text{nom}}) \triangleq \Delta A_p = -B_o \Delta k C_o, \tag{45}$$

where

$$B_o = \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}^T, \; C_o = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}.$$

We desire to design a constant gain linear feedback compensator of the form given in (12)-(13) such that the closed-loop system is stable for $0.5 < k < 2.0$ and for a unit impulse disturbance at $t = 0$, the performance variable $z_p$ has a settling time of about 15 s for the nominal system (with $k = k_{\text{nom}} = 1$).

### 6.1 $H_2$ Performance

The closed loop system is given by (14)-(15) and assuming negative feedback the state matrix $A$ is given by (16). The closed loop $B$ and $C$ matrices are given by

$$B = \begin{bmatrix} B_o & 0_{4\times 1} \end{bmatrix}, \quad C = \begin{bmatrix} C_o & 0_{1\times 4} \end{bmatrix} \tag{46}$$

Hence $G(s)$ and $\Delta$ in the uncertainty feedback configuration shown in Figure 1 are given by

$$G(s) \sim \left[ \begin{array}{c|c} A(k_{\text{nom}}) & B \\ \hline C & 0 \end{array} \right], \quad \Delta = \Delta k. \tag{47}$$

The upper bound on the $H_2$ cost functional is given by (7) in Theorem 1 where $V$ is given by

$$V = \begin{bmatrix} V_1 & V_{12} {B_c}^T \\ B_c V_{12}^T & B_c V_2 B_c^T \end{bmatrix}, \tag{48}$$

with $V_1 = D_1 {D_1}^T$, $V_2 = \rho D_2 {D_2}^T$, and $V_{12} = \sqrt{\rho}\, D_1 {D_2}^T$. and the weighting matrix $R$ is given by

$$R = \begin{bmatrix} R_1 & -R_{12} C_c \\ -C_c^T R_{12}^T & C_c^T R_2 C_c \end{bmatrix}, \tag{49}$$

where $R_1 = E_1^T E_1$, $R_2 = \rho$, and $R_{12} = \sqrt{\rho}\, E_1^T$. Here the parameter $\rho$ is used as a design parameter to increase or decrease the authority of the controller and is hence made a function of the homotopy parameter $\lambda$.

It can be seen that the diagonal $H$ and $N$ of the Popov multiplier, reduce to scalars for this particular example. The variable $x$ in (41) is given by

$$x = \begin{bmatrix} H & N & \epsilon & \text{vec}^T(A_c) & \text{vec}^T(B_c) & \text{vec}^T(C_c) \end{bmatrix}^T, \tag{50}$$

and consequently, the function $F$ in (42), is given by

$$F(x,\gamma) = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial H} \\ \frac{\partial \mathcal{L}}{\partial N} \\ \frac{\partial \mathcal{L}}{\partial \epsilon} \\ \text{vec}\left(\frac{\partial \mathcal{L}}{\partial A_c}\right) \\ \text{vec}\left(\frac{\partial \mathcal{L}}{\partial B_c}\right) \\ \text{vec}\left(\frac{\partial \mathcal{L}}{\partial C_c}\right) \end{bmatrix}. \tag{51}$$

The initial point $x_0$ is chosen in the following manner. $H_0$, $N_0$, and $\epsilon_0$, are chosen arbitrarily as 10, 10, and 1, respectively. The initial controller $(A_{c,0}, B_{c,0}, C_{c,0})$ is an LQG controller for the nominal plant corresponding to $\rho = 0.001$. No robustness is expected of this controller and hence $M_{1,0}$ and $M_{2,0}$ in (41) are chosen close to zero (i.e., $M_{1,0} = -0.01$ and $M_{2,0} = 0.01$). It is found that the controller authority need not be increased any more and hence $\rho$ is kept constant (i.e. $\rho_0 = \rho_f = 0.001$).

The controller transfer function obtained by the probability-one homotopy algorithm is given by

$$H(s) = \frac{2819\,(s+0.2079)[(s-0.0978)^2 + 0.8063^2]}{[(s+4.004)^2 + 1.8294^2][(s+3.4747)^2 + 9.9745^2]}. \tag{52}$$

This controller is guaranteed by the theory to be robust for the range $0.5 < k < 2.0$ and this was also verified by a direct search. The settling time for the system was chosen to be the time required for the displacement of mass 2 to reach and stay within the interval [-0.1m, 0.1m]. The controller is seen to satisfy the settling time objective when connected to the nominal model corresponding to $k = 1$ N/m, as can be seen from the impulse response of the closed-loop system in Figure 4.

### 6.2 $H_\infty$ Performance

As shown in Section 3, the problem may be formulated to minimize an $H_\infty$ instead of an $H_2$ performance index. The formulation is similar to that in [2]. The closed loop system may now be represented as

$$\dot{x} = (A - \tilde{B}\Delta\tilde{C})x \tag{53}$$

$$y = \tilde{C}x \tag{54}$$

where $A$, $\tilde{B}$ and $\tilde{C}$ are given by (16), (28) and (24).

We let the performance block $\Delta_p$ between $[z\ u]^T$ and $w = [w_1\ w_2]^T$ be diagonal, i.e.,

$$\Delta_p = \begin{bmatrix} \Delta_{p,1} & 0 \\ 0 & \Delta_{p,2} \end{bmatrix}. \tag{55}$$

This makes it easy to vary the bounds for $\Delta_{p,1}$ and $\Delta_{p,2}$ viz. $M_2(2,2)$ and $M_2(3,3)$ (with $M_1 = -M_2$), independently in order to change $z$ and $u$, respectively.

Hence $G(s)$ and $\Delta$ in Figure 1 is given by

$$G(s) \sim \left[ \begin{array}{c|c} A(k_{\text{nom}}) & \tilde{B} \\ \hline \tilde{C} & 0 \end{array} \right] \quad (56)$$

$$\Delta = \begin{bmatrix} \Delta k & 0 & 0 \\ 0 & \Delta_{p,1} & 0 \\ 0 & 0 & \Delta_{p,2} \end{bmatrix}, \quad M_1 < \Delta < M_2 \quad (57)$$

In this case $M_1$ and $M_2$ are diagonal matrices instead of scalars. Notice that in this formulation $\Delta$ is *mixed* since $\Delta k$ is real where as $\Delta_{p,1}$ and $\Delta_{p,2}$ are allowed to be complex. Hence multiplier elements $N(2,2)$ and $N(3,3)$ are constrained to be zero. Since $H$ and $N$ are no longer scalars the variable $x$ in (41) is given by

$$x = \begin{bmatrix} \text{vec}H \\ \text{vec}N \\ \epsilon \\ \text{vec}(A_c) \\ \text{vec}(B_c) \\ \text{vec}(C_c) \end{bmatrix}, \quad (58)$$

and consequently, the function $F$ in (42), is given by

$$F(x,\gamma) = \begin{bmatrix} \text{vec}\left(\frac{\partial \mathcal{L}}{\partial H}\right) \\ \text{vec}\left(\frac{\partial \mathcal{L}}{\partial N}\right) \\ \frac{\partial \mathcal{L}}{\partial \epsilon} \\ \text{vec}\left(\frac{\partial \mathcal{L}}{\partial A_c}\right) \\ \text{vec}\left(\frac{\partial \mathcal{L}}{\partial B_c}\right) \\ \text{vec}\left(\frac{\partial \mathcal{L}}{\partial C_c}\right) \end{bmatrix}. \quad (59)$$

The expressions for $F(x,\gamma)$ in this case are much more complex than in the case of $H_2$ performance since the $\tilde{B}$ and $\tilde{C}$ are now a function of controller parameters. Consider for example the expression for $\frac{\partial \mathcal{L}}{\partial B_c}$ for the $H_2$ and the $H_\infty$ controller given respectively by

$$\frac{1}{2}\frac{\mathcal{L}}{\partial B_c} = \frac{1}{\epsilon}P_{21}V_{12} + \frac{1}{\epsilon}[P_{22}B_cV_2] + [PQ]_{21}C_p^T$$

and

$$\frac{1}{2}\frac{\mathcal{L}}{\partial B_c} = [QP]_{12}^T + [M_1CQP]_{22}^T D_2^T$$

$$+[Y^{-1}H^2CQP]_{22}^T D_2^T - [NC]_{22}^T\{[Y^{-1}H^2CQPBY^{-1}]_{22}^T$$

$$+[Y^{-1}H^2CQPBY^{-1}]_{22}\}D_2^T - [QPBY^{-1}NC]_{12}^T C_p^T$$

$$[Y^{-1}NCAQP]_{22}^T D_2^T - [NC]_{22}^T\{[Y^{-1}NCAQPBY^{-1}]_{22}^T$$

$$+[Y^{-1}NCAQPBY^{-1}]_{22}\}D_2^T - [M_1CQPBY^{-1}NC]_{22}^T$$

$$-[Y^{-1}NCBM_1CQP]_{22}^T D_2^T$$

$$+[NC]_{22}^T\{[Y^{-1}NCBM_1CQPBY^{-1}]_{22}^T$$

$$+[Y^{-1}NCBM_1CQPBY^{-1}]_{22}\}D_2^T + \ldots$$

The expressions for the corresponding Jacobian are therefore even more complicated and lengthy. The numerical example for the robust controller with $H_\infty$ performance will be completed in the final paper.

## 7 Conclusions

In this paper the Popov Multiplier has been used to develop probability-one homotopy algorithms for the design of robust controllers with guaranteed $H_2/H_\infty$ performance. The formulation closely follows that presented in [3, 4] and extends it to the case of robust controllers with $H_\infty$ performance. Though the formulation for both the robust $H_2$ and the robust $H_\infty$ problems are very similar, the gradient and the Hessian expressions for the $H_\infty$ formulation are a lot more complex. A numerical benchmark example is presented for the robust $H_2$ controller. The numerical design of the robust $H_\infty$ controller and a comparison of these designs will be presented in the final paper.

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