Robust H_2 Estimation with Application to Robust Fault Detection

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Abstract

The key to estimator-based, robust fault detection is to generate residuals which are robust against plant uncertainties and external disturbance inputs, which in turn requires the design of robust estimators. Hence, this paper considers the design of robust H_2 estimators using a parameter-dependent bounding function approach in conjunction with multiplier theory (which is intimately related to mixed-structured singular value theory). Specifically, the Popov-Tsypkin multiplier is used to develop an upper bound on an H_2 cost function over an uncertainty set. The robust H_2 estimation problem is formulated as a parameter optimization problem in which the upper bound is minimized subject to a Riccati equation constraint. A continuation algorithm that uses quasi-Newton BFGS (the algorithm of Broyden, Fletcher, Goldfab, and Shanno) corrections is developed to solve the minimization problem. The robust H_2 estimation framework is then applied to the robust fault detection of dynamic systems. The results are applied to a simplified longitudinal flight control system. It is shown that the robust fault detection procedure based on the robust H_2 estimation methodology proposed in this paper can reduce false alarm rates.

Keywords: robust H_2 estimation, multiplier theory, mixed structured singular value, robust fault detection

Running Title: Robust H_2 Estimation and Fault Detection

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1. Introduction

Fault detection of dynamic systems has been an active research area in recent years.[1,5–7,11] Fault detection can be achieved by using either *physical redundancy* or *analytical redundancy* (e.g., estimator-based fault detection) methods. The key step in estimator-based fault detection methods is to generate residuals which are accentuated by faults. These residuals are then compared with some threshold values to determine whether faults have occurred. Logically, the existence of uncertainties and disturbance inputs (i.e., plant disturbances and measurement noise) obscures the effect of faults and is therefore a source of false alarms. In order to reduce false alarm rates and improve fault detection accuracy, the residuals generated should be robust against uncertainties and disturbance inputs. The residuals used in fault detection are generated by comparing the actual measurements of the plant with the corresponding estimated quantities which are obtained by estimation. The requirement of robust residual generation naturally leads to the problem of robust estimation which is the key factor in guaranteeing robust fault detection.

In this paper, the problem of robust H_2 estimation will be studied by using the parameter-dependent bounding function approach[3,8–10,12] in conjunction with multiplier theory,[2–4] which is less conservative than methods based on the small gain theorem and fixed Lyapunov function theory. A generic upper bound for the H_2 performance functional over an uncertainty set is developed by bounding the uncertain terms in a Lyapunov equation and specific structure is assigned to the bounding function by using the Popov-Tsypkin multiplier. The robust H_2 estimation problem is then formulated as a parameter optimization problem in which the upper bound is minimized subject to a Riccati equation constraint. A continuation algorithm that uses quasi-Newton (BFGS) corrections is developed to solve the minimization problem. The robust fault detection problem is then formulated using a robust H_2 estimation framework. A numerical example is presented to illustrate the design algorithms.

The paper begins in Section 2 with the formulation of the robust H_2 estimation problem for linear, discrete-time uncertain systems. Section 3 develops an upper bound for the H_2 performance functional over the uncertainty set by using the parameter-dependent bounding function approach. Section 4 casts the robust H_2 estimation problem as an auxiliary minimization problem and the algorithm developed to solve the robust H_2 estimation problem is discussed. Section 5 discusses the application of robust H_2 estimation to robust fault detection of dynamic systems with uncertainties and external disturbance inputs. Section 6 presents a numerical example and Section 7 concludes the paper.

Notation and Definitions

$\mathcal{R}, \mathcal{Z}^+$	real numbers, nonnegative integers
$\mathcal{R}^{m\times n}$	$m \times n$ real matrices
$\mathcal{D}^n$	$n \times n$ real diagonal matrix
$\mathcal{N}^n$	$n \times n$ nonnegative definite matrix
tr	trace
$M_2 > M_1$	$M_2 - M_1$ positive definite
$M_2 \geq M_1$	$M_2 - M_1$ nonnegative definite
$\mathbf{E}(\cdot)$	expectation
$\|G(z)\|_2$	$\left[\frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}(G(e^{j\theta}))^* G(e^{j\theta}))d\theta\right]^{1/2}$
$\dim(M)$	dimension of M
$\mathrm{Vec}(\cdot)$	the standard column stacking operator
$z_{i,j}$	(i,j) element of matrix Z

2. The Robust H_2 Estimation Problem

Consider the discrete-time, linear uncertain system

$$x_p(k+1) = (A_p + \Delta A_p)x_p(k) + D_{p,1}w_(k), k \in \mathcal{Z}^+ \tag{1}$$

$$y_p(k) = (C_p + \Delta C_p)x_p(k) + D_{p,2}w_(k) \tag{2}$$

where $x_p \in \mathcal{R}^{n_p}$ is the state vector, $y_p \in \mathcal{R}^{p_p}$ denotes the plant measurements, $w(k) \in \mathcal{R}^{d_p}$ is a zero mean white noise signal satisfying $\mathbf{E}(w(k)w(\ell)^T) = \delta(k,\ell)I,\ D_{p,1}D_{2,p}^T = 0$ and the uncertainties ΔA_p and ΔC_p satisfy

$$\Delta A_p \in \mathcal{U}_{A_p} \triangleq \{\Delta A_p \in \mathcal{R}^{n_p\times n_p} : \Delta A_p = -B_{A_p}F_{A_p}C_{A_p},\ F_{A_p} \in \mathcal{F}_{A_p}\}, \tag{3}$$

$$\Delta C_p \in \mathcal{U}_{C_p} \triangleq \{\Delta C_p \in \mathcal{R}^{p_p\times n_p} : \Delta C_p = -B_{C_p}F_{C_p}C_{C_p},\ F_{C_p} \in \mathcal{F}_{C_p}\}, \tag{4}$$

where

$$\mathcal{F}_{A_p} \triangleq \{F_{A_p} \in \mathcal{D}^r : M_{A_p,1} \leq F_{A_p} \leq M_{A_p,2}\}, \tag{5}$$

$$\mathcal{F}_{C_p} \triangleq \{F_{C_p} \in \mathcal{D}^s : M_{C_p,1} \leq F_{C_p} \leq M_{C_p,2}\}, \tag{6}$$

with $M_{A_p,1},\ M_{A_p,2} \in \mathcal{D}^r,\ M_{C_p,1},\ M_{C_p,2} \in \mathcal{D}^s,\ M_{A_p,2} - M_{A_p,1} \geq 0$, and $M_{C_p,2} - M_{C_p,1} \geq 0$. It is desired to design a predictive filter of the form

$$x_e(k+1|k) = A_e x_e(k|k-1) + W[y_p(k) - C_p x_e(k|k-1)] \tag{7}$$

to estimate (in some sense to be defined) the state vector x_p, where $A_e \in \mathcal{R}^{n_p\times n_p}$ and $W \in \mathcal{R}^{n_p\times p_p}$ are the filter parameters to be determined.

Define the estimation error as

$$e(k) \triangleq x_p(k) - x_e(k|k-1), \tag{8}$$

which using (1), (2), and (7) can be shown to obey the evolution equation

$$e(k+1) = (A_e - WC_p)e(k) + (A_p + \Delta A_p - W\Delta C_p - A_e)x_p(k) + (D_{p,1} - WD_{p,2})w(k). \tag{9}$$

Next, define the error output $z \in \mathcal{R}^{q_p}$ as $z(k) \triangleq E_p e(k)$. Then augmenting (1) with (9) yields

$$x(k+1) = (A + \Delta A)x(k) + Dw(k) \tag{10}$$

$$z(k) = Ex(k) \tag{11}$$

where

$$x(k) = \begin{bmatrix} x_p(k) \\ e(k) \end{bmatrix}, A = \begin{bmatrix} A_p & 0 \\ A_p - A_e & A_e - WC_p \end{bmatrix}, \tag{12}$$

$$D = \begin{bmatrix} D_{p,1} \\ D_{p,1} - WD_{p,2} \end{bmatrix}, \qquad E = \begin{bmatrix} 0 & E_p \end{bmatrix}. \tag{13}$$

Furthermore, ΔA satisfies

$$\Delta A \in \mathcal{U}_A \triangleq \{\Delta A \in \mathcal{R}^{2n_p \times 2n_p} : \ \Delta A = -B_0 F_A C_0, \ F_A \in \mathcal{F}_A\}, \tag{14}$$

$$\mathcal{F}_A \triangleq \{F_A \in \mathcal{D}^{r+s} : \ M_1 \le F_A \le M_2\}, \tag{15}$$

where

$$F_A \triangleq \begin{bmatrix} F_{A_p} & 0 \\ 0 & F_{C_p} \end{bmatrix}, \quad B_0 = \begin{bmatrix} B_{A_p} & 0 \\ B_{A_p} & WB_{C_p} \end{bmatrix}, \quad C_0 = \begin{bmatrix} C_{A_p} & 0 \\ -C_{C_p} & 0 \end{bmatrix}, \tag{16}$$

and

$$M_1 = \mathrm{diag}(M_{A_p,1}, M_{C_p,1}), \quad M_2 = \mathrm{diag}(M_{A_p,2}, M_{C_p,2}). \tag{17}$$

The Robust H_2 Estimation Problem. Find A_e and W such that

1. the augmented system (10) is asymptotically stable over the uncertainty set $\mathcal{U}_A$; and

2. the H_2 performance functional

$$\mathcal{J}(A_e, W) = \sup_{\Delta A \in \mathcal{U}_A} \|G_{zw}\|_2 \tag{18}$$

is minimized, where G_{zw} is the transfer function matrix from the disturbance w to the performance variable z.

3. Bounds on H_2 Performance Over the Uncertainty Set

In this section, a generic upper bound on the H_2 performance functional (18) will be developed over the entire uncertainty set for an uncertain system by using a parameter-dependent bounding function approach similar to that proposed by Haddad and Bernstein.[8,9] Then specific structure is assigned to the bounding function by using the Popov-Tsypkin multiplier.[3,10,12] The upper bound to be developed can be used in the synthesis of robust H_2 estimators.

For the uncertain discrete-time linear system

$$x(k+1) = (A+\Delta A)x(k) + Dw(k), \tag{19}$$

$$z(k) = Ex(k), \tag{20}$$

where for some uncertainty set $\mathcal{U} \subset \mathcal{R}^{n\times n}$, $\Delta A \in \mathcal{U}$, $x \in \mathcal{R}^n$, $z \in \mathcal{R}^q$, and $w(\cdot) \in \mathcal{R}^d$ denotes a white noise disturbance signal, the following proposition provides a characterization of the H_2 performance functional, $\sup_{\Delta A\in\mathcal{U}} \|G_{zw}\|_2$.

Proposition 1. Suppose $A+\Delta A$ is asymptotically stable for all $\Delta A \in \mathcal{U}$. Then

$$\mathcal{J}(\mathcal{U}) \triangleq \sup_{\Delta A\in\mathcal{U}} \|G_{zw}\|_2 = \sup_{\Delta A\in\mathcal{U}} \mathrm{tr} Q_{\Delta A} R, \tag{21}$$

where $Q_{\Delta A}$ is generated by

$$Q_{\Delta A} = (A+\Delta A)Q_{\Delta A}(A+\Delta A)^T + V, \tag{22}$$

and $R \triangleq E^T E$, $V \triangleq DD^T$.

Proof. See References 8 and 9. □

In the following theorem, an upper bound on the H_2 performance functional (21) is obtained by bounding $Q_{\Delta A}$ for $\Delta A \in \mathcal{U}$ by means of a parameter-dependent bounding function. This theorem forms the foundation for later developments.

Theorem 1. Let $\Omega_0 : \mathcal{N}^n \to \mathcal{S}^n$ and $Q_{0,\Delta A} : \mathcal{U} \to \mathcal{N}^n$ be such that

$$\begin{aligned} &\Delta A Q A^T + AQ\Delta A^T + \Delta A Q \Delta A^T \le \Omega_0(Q) - \\ &[(A+\Delta A)Q_{0,\Delta A}(A+\Delta A)^T - Q_{0,\Delta A}], \quad \Delta A \in \mathcal{U}, \quad Q \in \mathcal{N}^n. \end{aligned} \tag{23}$$

Also, suppose there exists $Q \in \mathcal{N}^n$ satisfying

$$Q = AQA^T + \Omega_0(Q) + V, \tag{24}$$

and such that $Q + Q_{0,\Delta A}$ is nonnegative definite for all $\Delta A \in \mathcal{U}$. Then

$$(A + \Delta A) \text{ is detectable}, \quad \Delta A \in \mathcal{U}, \tag{25}$$

if and only if

$$A + \Delta A \text{ is asymptotically stable}, \quad \Delta A \in \mathcal{U}. \tag{26}$$

In this case,

$$Q_{\Delta A} \le Q + Q_{0,\Delta A}, \quad \Delta A \in \mathcal{U}, \tag{27}$$

where $Q_{\Delta A}$ is given by (22). If, in addition, there exists $\bar{Q}_0 \in \mathcal{N}^n$ such that

$$Q_{0,\Delta A} \le \bar{Q}_0, \quad \Delta A \in \mathcal{U}, \tag{28}$$

then

$$\mathcal{J}(\mathcal{U}) \le \operatorname{tr}[Q + \bar{Q}_0]R. \tag{29}$$

Proof. The proof is a specialization of the proof of the corresponding theorem in Reference 12 for finite-horizon cost functionals. □

Now define the uncertainty set $\mathcal{U}_A \in \mathcal{R}^{n\times n}$ as

$$\mathcal{U}_A \triangleq \{\Delta A \in \mathcal{R}^{n\times n} : \Delta A = -B_0 F C_0, F \in \mathcal{F}\}, \tag{30}$$

where $\mathcal{F}$ satisfies

$$\mathcal{F} \triangleq \{F \in \mathcal{D}^m : M_1 \le F \le M_2\}. \tag{31}$$

Next, a specific H_2 upper bound is developed for the H_2 performance functional (21) by using Theorem 1 and the Popov-Tsypkin multiplier[3,10,12] which can be written in the transfer function form as

$$M(z) = H + N\frac{z-1}{z}, \tag{32}$$

where $H \in \mathcal{D}^n, N \in \mathcal{D}^n$. The upper bound to be developed provides the theoretical basis for characterization of a solution to the robust H_2 estimation problem.

Augmenting multiplier (32) with the uncertain system $G(z) \sim (A, B_0, C_0, 0)$ yields $M(z)G(z)$ which has the following state space realizations:

$$A_a = \begin{bmatrix} 0 & 0 \\ B_0 & A \end{bmatrix}, \quad D_a = \begin{bmatrix} 0 \\ D \end{bmatrix}, \quad E_a = \begin{bmatrix} 0 & E \end{bmatrix}, \tag{33}$$

and

$$\Delta A_a \in \mathcal{U}_a \triangleq \{\Delta A_a \in \mathcal{R}^{m+n} : \Delta A_a = -B_a F C_a,\ F \in \mathcal{F}\}, \tag{34}$$

where

$$B_a = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \qquad C_a = \begin{bmatrix} I & C_0 \end{bmatrix}. \tag{35}$$

In terms of the augmented system, the H_2 performance functional (21) can be written as

$$\mathcal{J}(\mathcal{U}) \leq \mathcal{J}(\mathcal{U}_a) \triangleq \sup_{\Delta A_a \in \mathcal{U}_a} \mathrm{tr} Q_{a,\Delta A} R_a, \tag{36}$$

where $Q_{a,\Delta A}$ is generated by

$$Q_{a,\Delta A} = (A_a + \Delta A_a) Q_{a,\Delta A} (A_a + \Delta A_a)^T + V_a, \quad \Delta A_a \in \mathcal{U}_a, \tag{37}$$

and

$$R_a \triangleq E_a^T E_a, \qquad V_a \triangleq D_a D_a^T. \tag{38}$$

The next theorem provides an upper bound for $\mathcal{J}(\mathcal{U}_a)$ in (36). Before presenting the theorem, define $S_a = \begin{bmatrix} I \\ 0 \end{bmatrix}$, where $\dim(S_a) = \dim(B_a)$.

Theorem 2. Suppose there exists diagonal $H \in \mathcal{D}^n$, $N \in \mathcal{D}^n$, and $Q_a \geq 0$ satisfy $2H(M_2 - M_1)^{-1} - C_a Q_a C_a^T > 0$ and

$$\begin{aligned} Q_a \;=\; & (A_a - B_a M_1 C_a) Q_a (A_a - B_a M_1 C_a)^T \\ +\; & [(A_a - B_a M_1 C_a) Q_a C_a^T - B_a (H + N) + S_a N] \cdot [2H(M_2 - M_1)^{-1} - C_a Q_a C_a^T]^{-1} \cdot \\ & [(A_a - B_a M_1 C_a) Q_a C_a^T - B_a (H + N) + S_a N]^T + V_a. \end{aligned} \tag{39}$$

Then

$$(A_a + \Delta A_a, E_a) \text{ is detectable}, \quad \Delta A_a \in \mathcal{U}_a, \tag{40}$$

if and only if

$$A_a + \Delta A_a \text{ is asymptotically stable}, \quad \Delta A_a \in \mathcal{U}_a. \tag{41}$$

Furthermore,

$$\mathcal{J}(\mathcal{U}_a) \leq \mathrm{tr} Q_a R_a. \tag{42}$$

Proof. Refer to Reference 12.

Remark 1. Although we have explicitly only considered real parametric uncertainty, the results can be easily modified to the more general case of mixed (i.e., real parametric and complex) uncertainty. This modification requires that the elements of N corresponding to the complex uncertainty be zero.

4. An Algorithm for Robust H_2 Estimation

In this section, the general results presented in Section 3 will be used to solve the robust H_2 estimation problem posed in Section 2. Based on Theorem 2, the robust H_2 estimation problem can be formulated as an auxiliary minimization problem in which the H_2 cost upper bound $J(A_e, W)$, given by

$$\mathcal{J}(A_e, W) \leq J(A_e, W) \triangleq \mathrm{tr} Q_a R_a, \tag{43}$$

is minimized subject to the Riccati equation constraint

$$\begin{aligned} Q_a \;=\; & (\tilde{A}_a + BWB_c + BA_eC)Q_a(\tilde{A}_a + BWB_c + BA_eC)^T + \\ & [(\tilde{A}_a + BWB_c + BA_eC)Q_aC_a^T - (\tilde{B}_a + BWC_B)(H+N) + \\ & S_aN] \cdot Y^{-1} \cdot [(\tilde{A}_a + BWB_c + BA_eC)Q_aC_a^T - \\ & (\tilde{B}_a + BWC_B)(H+N) + S_aN]^T + V_a, \end{aligned} \tag{44}$$

where

$$A_a - B_aM_1C_a = \begin{bmatrix} 0 & 0 \\ B_0 & A - B_0M_1C_0 \end{bmatrix} \triangleq \tilde{A}_a + BWB_c + BA_eC, \tag{45}$$

$$B_a = \begin{bmatrix} 0 \\ B_0 \end{bmatrix} \triangleq \tilde{B}_a + BWC_B, \quad B_c \triangleq \begin{bmatrix} 0 & B_{C_p} & B_{C_p}M_{C_p,1}C_{C_p} & -C_p \end{bmatrix}, \tag{46}$$

$$C_B \triangleq \begin{bmatrix} 0 & B_{C_p} \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix}^T, \quad C \triangleq \begin{bmatrix} 0 & 0 & -I & I \end{bmatrix}, \quad S_a \triangleq \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{47}$$

$$V_1 \triangleq D_{p,1}D_{p,1}^T, \qquad V_2 \triangleq D_{p,2}D_{p,2}^T, \tag{48}$$

$$V_0 \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & V_1 & V_1 \\ 0 & 0 & V_1 & V_1 \end{bmatrix}, \qquad V_a \triangleq V_0 + BWV_2W^TB^T, \tag{49}$$

$$Y \triangleq 2H(M_2 - M_1)^{-1} - C_aQ_aC_a^T. \tag{50}$$

Additional details of the construction of this auxiliary minimization problem are given in Reference 12. To solve this optimization problem, form the Lagrangian

$$\mathcal{L}(P, Q_a, W, A_e, H, N) = \mathrm{tr} Q_a R_a + \mathrm{tr}\{P[-Q_a + \text{the R.H.S. of (44)}]\}, \tag{51}$$

where $P \geq 0$ is the Lagrangian multiplier.

Algorithm for Solving the Robust H_2 Estimation Problem

The following algorithm uses a continuation parameter $\lambda \in [0,1)$ which modifies the uncertainty bounds M_1 and M_2 at each iteration such that

$$M_1 = M_1(\lambda) \triangleq M_{10} + \lambda(M_{1f} - M_{10}),$$

$$M_2 = M_2(\lambda) \triangleq M_{20} + \lambda(M_{2f} - M_{20}).$$

Note that $M_1(0) = M_{10},\ M_1(1) = M_{1f},\ M_2(0) = M_{20}$, and $M_2(1) = M_{2f}$. In practice, M_{10} and M_{20} are assigned very small values. M_{1f} and M_{2f} equal, respectively, the desired lower and upper bounds of the uncertainty set. In addition,

$$\theta \triangleq \begin{bmatrix} \mathrm{Vec}(W)^T & \mathrm{Vec}(A_e)^T & h_{11} & h_{22} & \ldots & h_{r+s,r+s} & n_{11} & n_{22}, & \ldots & n_{r+s,r+s} \end{bmatrix}^T,$$

and $\mathcal{H}$ denotes the current estimate of the Hessian matrix, $\frac{\partial^2 J}{\partial \theta^2} = \frac{\partial^2 \mathcal{L}}{\partial \theta^2}$.

1. Let $\lambda = 0$. Let θ_0 be defined such that W and A_e are initialized with Kalman filter gains, the multiplier matrices H and N are initialized by solving an LMI. Let $\mathcal{H}_0^{-1} = I$.

2. For $k = 0, 1, 2, \ldots$, do

   (a) Determine a search direction $p_k = -\mathcal{H}_k^{-1}\frac{\partial J(\theta_k)}{\partial \theta_k}$, where $\frac{\partial J(\theta_k)}{\partial \theta_k} = \frac{\partial \mathcal{L}}{\partial \theta_k}$.

   (b) Use one dimensional line search to determine the step length η_k that minimizes $J(\theta_k + \eta_k p_k)$ with respect to η_k.

   (c) Set $\theta_{k+1} = \theta_k + \eta_k p_k$.

   (d) If $|\frac{\partial J(\theta_{k+1})}{\partial \theta_{k+1}}| < \epsilon^*$, where ϵ^* is a small number, go to step 3;
   Else, let $k = k+1$, update $\mathcal{H}_k^{-1}$, using the BFGS inverse Hessian update (Fletcher 1987), and go to step (a).

3. If $\lambda = 1$, let $\theta^* = \theta_{k+1}$ and stop where θ^* is the final solution;
   Else, set $\lambda = \lambda + \Delta\lambda,\ \theta_0 = \theta_{k+1},\ \mathcal{H}_0^{-1} = \mathcal{H}_{k+1}^{-1}$ and go to step 2.

5. Robust Fault Detection

In this section, the robust H_2 estimation framework presented in the previous sections is applied to robust fault detection for uncertain dynamic systems with white noise disturbance inputs. Consider the uncertain discrete-time system

$$x_p(k+1) = (A_p + \Delta A_p)x_p(k) + D_{p,1}w(k) + R_{p,1}f(k), \tag{52}$$

$$y_p(k) = C_p x_p(k) + D_{p,2}w(k) + R_{p,2}f(k) \tag{53}$$

where x_p, y_p, and w are as discussed in previous sections, and $f \in \mathcal{R}^{n_f}$ is the fault vector. The term $R_{p,1}f(k)$ represents actuator and component faults while $R_{p,2}f(k)$ denotes the sensor faults. The fault distribution matrices $R_{p,1}$ and $R_{p,2}$ are assumed to be known.

The robust fault detection problem is to generate a robust residual signal $r(k)$ that satisfies

$$\gamma(r(k)) \leq J_{th} \quad \text{if} \quad f(k) = 0, \tag{54}$$

$$\gamma(r(k)) > J_{th} \quad \text{if} \quad f(k) \neq 0, \tag{55}$$

where $\gamma(r(k))$ is some measure of the size of the residual, e.g., a norm, and J_{th} is a threshold value. In this paper, the variance of $r(k)$ is used instead of a norm since this is a more natural measure for H_2 estimation. The residual generated is given by the following equation if estimator (7) is applied to the system described by (52) and (53).

$$\begin{aligned} r(k) &= y_p(k) - C_p x_e(k) \qquad (56) \\ &= C_p e(k) + D_{p,2} w(k) + R_{p,2} f(k), \end{aligned}$$

where the estimation error $e(k)$ is governed by

$$\begin{aligned} e(k+1) &= (A_{e,k} - WC_p)e(k) + (\Delta A_p + A_p - A_{e,k})x_p(k) \\ &+ (D_{p,1} - WD_{p,2})w(k + (R_{p,1} - WR_{p,2})f(k). \qquad (57) \end{aligned}$$

It is clear from equations (11)-(13) and (56) that if E_p is chosen as C_p, then

$$r(k) = z(k) + D_{p,2}w(k) + R_{p,2}f(k). \tag{58}$$

In equation (58), z is the state estimation error. If no faults occur, such that $f(k) \triangleq 0$, and we define $var(\cdot)$ for a stochastic vector signal ξ of zero mean by

$$\text{var}(\xi) \triangleq \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} \mathbf{E}[\xi(k)\xi(k)^T], \tag{59}$$

then it follows that

$$\text{var}(r) = \text{var}(z) + \text{var}(D_{p,2}w). \tag{60}$$

But since,

$$\text{var}(z) \leq \text{tr}Q_a R_a, \qquad \text{var}(D_{p,2}w) = \text{tr}V_2, \tag{61}$$

(60) yields

$$\text{var}(r) \leq \text{tr}Q_a R_a + \text{tr}V_2 \triangleq J_{th}. \tag{62}$$

In practice, we do not use the steady-state variance of the residual signal (59) since we would like to identify faults as quickly as possible, but instead make the approximation

$$\frac{1}{N - N_0} \|r(k)\|_{[N_0,N]} \cong \text{var}(r(k)) \tag{63}$$

where

$$\|r(k)\|_{[N_0,N]} \triangleq \sum_{k=N_0}^{N} r(k)r(k)^T. \tag{64}$$

6. Illustrative Example

A numerical example is presented in this section to illustrate robust H_2 estimator design using the Popov-Tsypkin multiplier and the application of the robust H_2 estimator to robust fault detection of dynamic systems. The example used is a linearized discrete-time model of a simplified longitudinal flight control system[1] which is characterized by

$$\begin{aligned} A_p &= \begin{bmatrix} 0.8950 & -0.1083 & -0.3872 \\ 0.0015 & 0.8912 & -0.0672 \\ 0 & 0.7368 & 0 \end{bmatrix}, \qquad C_p = I_{3\times3}, \\ D_{p,1} &= \text{diag}\{0.1,\ 0.1,\ 0.01\}, \qquad D_{p,2} = 0.1 \times I_{3\times3}. \end{aligned}$$

The uncertainty matrix $\Delta A_p = -B_{A_p}F_{A_p}C_{A_p}$, where

$$\begin{aligned} B_{A_p} &= -\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad C_{A_p} = \begin{bmatrix} I_{3\times3} \\ I_{3\times3} \end{bmatrix}, \\ F_{A_p} &= \text{diag}\{\delta_1,\ \delta_2,\ \delta_3,\ \delta_4,\ \delta_5,\ \delta_6\} \end{aligned}$$

with $-0.0537 \le \delta_1 \le 0.0537$, $-0.0065 \le \delta_2 \le 0.0065$, $-0.0232 \le \delta_3 \le 0.0232$, $-0.00009 \le \delta_4 \le 0.00009$, $-0.0535 \le \delta_5 \le 0.0535$, and $-0.0041 \le \delta_6 \le 0.0041$. Note that the uncertain parameters δ_1 through δ_6 correspond to $\pm 6\%$ parameter fluctuations in the first two rows of matrix A_p.

For this particular example, the error output is defined as $z(k) = E_p e(k)$ where $E_p \triangleq I_{3\times3} = C_p$ and hence $z(k)$ may be used in the residual equation (58). The robust H_2 estimation problem is to design A_e and W in (7) such that the uncertain system (10) is asymptotically stable for each ΔA in the uncertainty set and the upper bound (43) on H_2 performance is minimized.

Kalman filter gains are used as the initial estimator gains for the robust H_2 estimation algorithm. After an LMI test using the initializing Kalman filter, the feasible initial values of the multiplier matrices are obtained as $H = 0.3946 \times I_{9\times9}$ and $N = 2.0895 \times I_{9\times9}$. Next using this estimator gain W and $A_e = A_p$ as the initial estimator parameters together with the initial multiplier matrices, robust H_2 estimator gains are computed by using the continuation algorithm.

In order to illustrate the application of the robust H_2 estimator to the robust fault detection of the system subject to plant uncertainties and disturbances, the uncertain parameters $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\}$ are assigned their upper bounds and uncorrelated white noise signals $w_1(k), w_2(k)$, and $w_3(k)$, with respective noise variances of 2, 1, and 1 are added as the disturbance inputs to

Figure 1: Robust Fault Detection vs. Nonrobust Fault Detection

the system. The sampling period used is 0.01 sec. and the time interval used is $N - N_0 = 20$. In order to evaluate the performance of both the Kalman filter and the robust H_2 estimator in the presence of a system fault, a sensor fault is added to the velocity sensor at the time instant $t = 3$ sec. Specifically, it is assumed that the *faulty* sensor's reading is 0.5 times the actual system velocity. Figure 1 shows the residuals generated by using both the Kalman filter and the robust H_2 estimator. Note that the measure of the residual corresponding to the robust estimator is almost zero. It can be seen that both filters can detect this fault because the variances of both residuals surpass their respective threshold values after $t = 3$ sec.. However, even prior to $t = 3$ sec., the residual variance generated by Kalman filter surpasses the threshold value even though there is no fault in the system and thus issues a false alarm.

7. Conclusion

In this paper, the problem of robust H_2 estimation for uncertain, linear discrete-time systems and its applications to the robust fault detection of dynamic systems has been addressed. An upper bound for the H_2 performance functional has been developed by using a parameter-dependent bounding function approach in conjunction with the Popov-Tsypkin multiplier. The robust H_2 estimation problem was formulated as a constrained parameter estimation problem. A continuation algorithm was developed to synthesize the estimator. The robust H_2 estimation framework was then used in estimator-based fault detection of dynamic systems. By considering a flight longitudinal system, it was shown that the robust fault detection methodology based on the robust H_2 estimation framework is capable of significantly reducing false alarm rates.

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