

# Robust Controller Synthesis via Nonlinear Matrix Inequalities

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## Abstract

Over the last several years fixed-structure multiplier versions of MSSV theory have been developed and have led to the development of LMI's for the analysis of robust stability and performance. These LMI's have in turn led to the development of BMI's for the synthesis of robust controllers. The BMI formulation in practice requires the multiplier to lie in the span of a stable basis, potentially introducing significant conservatism. This paper uses the LMI approach to MSSV analysis to develop an approach to robust controller synthesis that is based on the stable factors of the multipliers and does not require the multipliers to be restricted to a basis. It is shown that this approach requires the solution of nonlinear matrix inequalities. A continuation algorithm is presented for the solution of NMI's. The primary computational burden of the continuation algorithm is the solution of a series of LMI's.

**Keywords:** Fixed-structure multipliers, multivariable stability margin, LMI's, BMI's, continuation algorithms

**Running Title:** Robust Controller Synthesis via Nonlinear Matrix Inequalities

## 1. Introduction

Mixed structured singular value (MSSV) theory was developed to nonconservatively analyze the robust stability and performance of systems with both real parametric and complex uncertainty [12, 43]. The original results [12, 43] were stated in terms of frequency dependent multipliers. However, subsequently fixed-structure multiplier versions of MSSV analysis were developed independently

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in [17, 18, 19, 22, 23, 24, 25] and [6, 29, 35]. The latter research soon led to LMI formulations of MSSV robustness tests [29, 4].

A unique contribution of the former results is that they led to the development of upper bounds on an  $H_2$  cost functional over the uncertainty set under consideration. By optimizing this upper bound and using a Riccati equation, constraint continuation algorithms have been developed for MSSV synthesis [26, 27, 28] with the Popov multiplier. It should be noted that this approach allows the control designer to fix the architecture of the controller (e.g., the controller may be chosen to have a particular order or may be chosen to have a decentralized structure) and avoids  $M$ - $K$  (i.e., multiplier-controller) iteration schemes. Recently, more powerful probability-one homotopy algorithms have been formulated for more general forms of the multiplier by solving a “Riccati Equation Feasibility Problem” [9, 10]. (The multipliers are actually represented by their stable factors.) These algorithms, unlike the previous continuation algorithms which were initialized by an ad hoc technique, can be initialized with an arbitrary (admissible) multiplier and any stabilizing compensator (of the desired architecture). In addition, as with all probability-one algorithms, the homotopy zero curve is not assumed to be monotonic which makes the algorithms more reliable and numerically robust.

The LMI formulations of MSSV theory led to the recognition that robust control design can be approached via solving a (nonconvex) “bilinear matrix inequality” (BMI) [15, 16, 36]. This approach, like those based on a Riccati equation constraint, allows the design of fixed-architecture controllers and can be implemented without using  $M$ - $K$  iteration. To obtain a reasonably sized BMI, the multiplier set must be restricted to lie in the span of a stable basis [15]. However, the choice of this basis is unclear and can potentially introduce a high degree of conservatism. If the less conservative LMI formulation, requiring the use of unstable multipliers, is used, the resultant BMI is of very high dimension due to the introduction of a Lyapunov inequality of the dimension of the closed-loop system to ensure closed-loop stability [36]. In contrast, the robustness analysis results using a Riccati equation formulation easily extend to robust control design without placing any basis restrictions on the multipliers or introducing high dimensionality.

This paper uses the LMI approach to MSSV analysis to develop an approach to robust controller synthesis that is based on the stable factors of the multipliers and does not require the multipliers to be restricted to a basis. It is shown that this approach requires the solution of nonlinear matrix inequalities. A continuation algorithm is presented for the solution of NMI’s. The primary computational burden of the continuation algorithm is the solution of a series of LMI’s.

## 1.1. Notation and Definitions

Let  $\mathcal{R}$  and  $\mathcal{C}$  denote the real and complex numbers,  $\mathcal{R}^{m \times m}$  and  $\mathcal{C}^{m \times m}$  the real and complex  $m \times m$  matrices, let  $(\cdot)^T$  and  $(\cdot)^*$  denote transpose and complex conjugate transpose, let “Re” and “Im” denote real and imaginary part, and let  $I_n$  or  $I$  denote the  $n \times n$  identity matrix. Furthermore we write  $\sigma_{\max}(\cdot)$  for the maximum singular value, “tr” for the trace operator, and  $M \geq 0$  ( $M > 0$ ) to denote the fact that the Hermitian matrix  $M$  is nonnegative (positive) definite. The Hermitian and skew-Hermitian parts of an arbitrary complex square matrix  $G$  are defined by  $\text{He } G \triangleq \frac{1}{2}(G + G^*)$  and  $\text{Sh } G \triangleq \frac{1}{2}(G - G^*)$ , respectively. Finally,

Next, we establish certain key definitions used later in the paper. Let  $n(s)$  and  $d(s)$  be polynomials in  $s$  with real coefficients. A function  $g(s)$  of the form  $g(s) = n(s)/d(s)$  is called a *real rational function*. The function  $g(s)$  is called *proper* (resp., *strictly proper*) if  $\deg n(s) \leq \deg d(s)$  (resp.,  $\deg n(s) < \deg d(s)$ ), where “deg” denotes the degree of the respective polynomials. A *real-rational matrix function* is a matrix whose elements are rational functions with real coefficients. Furthermore, a *transfer function*  $G(s)$  is called *proper* (resp., *strictly proper*) if every element of  $G(s)$  is proper (resp., strictly proper). In this paper we assume all transfer functions are real-rational matrix functions. Also, we define  $G^*(s) \triangleq G^T(-s)$  for transfer functions  $G(s)$ .

An *asymptotically stable transfer function* is a transfer function each of whose poles is in the open left half plane. Finally, a *Lyapunov stable transfer function* is a transfer function each of whose poles is in the closed left half plane with semi-simple poles on the imaginary axis. Let  $G(s) \sim \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$  denote a state space realization of a transfer function  $G(s)$ , that is,  $G(s) = C(sI - A)^{-1}B + D$ .

A square transfer function  $G(s)$  is called *positive real* (resp., *generalized positive real*) [2] if  $G(s)$  is Lyapunov stable and  $\text{He } G(s)$  is nonnegative definite for  $\text{Re}[s] > 0$  (resp.,  $G(s)$  has no imaginary poles and  $\text{He } G(j\omega)$  is nonnegative definite for all  $\omega \in \mathcal{R}$ ). A square transfer function is called *strictly positive real* [42] (resp., *strictly generalized positive real*) if  $G(s)$  is asymptotically stable and  $\text{He } G(j\omega)$  is positive definite for  $\omega \in \mathcal{R}$  (resp.,  $G(s)$  has no imaginary poles and  $\text{He } G(j\omega)$  is positive definite for  $\omega \in \mathcal{R}$ ). A square transfer function  $G(s)$  is *strongly positive real* (resp., *strongly generalized positive real*) if it is strictly positive real (resp., strictly generalized positive real) and  $D + D^T > 0$ , where  $D \triangleq G(\infty)$ . Note that although a minimal realization of a positive real transfer function is stable in the sense of Lyapunov, a minimal realization of a generalized positive real transfer function may be unstable.

Let  $F : \mathcal{R}^n \rightarrow \mathcal{R}^{p \times p}$  and consider the inequalities

$$F(x) \geq 0, \quad F(x) > 0. \tag{1.1}$$

If  $F(x)$  is a linear function, then (1.1) are *linear matrix inequalities* (LMI's). If  $F(x)$  is affine, then then (1.1) are *affine matrix inequalities* (AMI's). However, since each AMI is equivalent to a certain LMI [5], AMI's are popularly called LMI's. If  $F(x)$  is nonlinear then (1.1) are *nonlinear matrix inequalities* (NMI's). Furthermore, if we partition  $x$  as  $x^T = (x_1^T, x_2^T)$  and  $F(x)$  is affine in  $x_1$  and  $x_2$  but not in  $x$ , then (1.1) are *bilinear matrix inequalities* (BMI's). Note that a BMI is a special case of a NMI.

## 1.2. Paper Organization

Section 2 presents the general framework for robustness analysis with fixed-structure multipliers and briefly discusses the BMI approach to fixed-architecture, robust control synthesis. Section 3 demonstrates that a more general formulation to robust controller synthesis is in terms of a NMI feasibility problem. Section 4 develops a continuation algorithm to solve NMI feasibility problems.

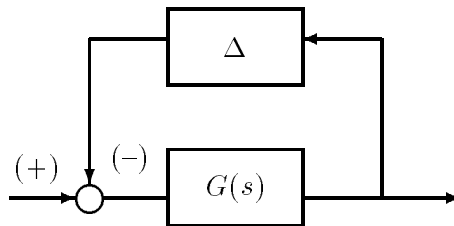


Figure 1: Standard Uncertainty Feedback Configuration

## 2. Multiplier Methods in Robustness Analysis

In this section we review the framework for mixed uncertainty robustness analysis with fixed-structure multipliers. The exposition generally follows that presented in References 24,29-4. We begin by considering the standard uncertainty feedback configuration of Figure 1, where  $G(s) \in \mathcal{C}^{m \times m}$  is asymptotically stable and  $G(s) \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ . It is assumed that the uncertainty  $\Delta \in \mathcal{C}^{m \times m}$  belongs to the set

$$\Delta_\gamma \triangleq \{\Delta = \text{block-diag}(\Delta_1, \dots, \Delta_p) : \Delta_i \in \mathcal{I}_i, \sigma_{\max}(\Delta_i) \leq \gamma, i = 1, \dots, p, \sum_{i=1}^p k_i = m\}, \quad (2.1)$$

where  $\mathcal{I}_i \subseteq \mathcal{C}^{k_i \times k_i}$  denotes the internal structure of the uncertainty block  $\Delta_i$  and  $\gamma > 0$ . For example  $\mathcal{I}_i$  may be given by any of the following five sets:

$$\Delta^{\text{I}} \triangleq \mathcal{C}^{k_i \times k_i}, \quad (2.2)$$

$$\Delta^{\text{II}} \triangleq \{\Delta_i = \delta_i I_{k_i} : \delta_i \in \mathcal{C}\}, \quad (2.3)$$

$$\Delta^{\text{III}} \triangleq \{\Delta_i = \delta_i I_{k_i} : \delta_i \in \mathcal{R}\}, \quad (2.4)$$

$$\Delta^{\text{IV}} \triangleq \{\Delta_i \in \mathcal{R}^{k_i \times k_i} : \Delta_i = \Delta_i^T\}, \quad (2.5)$$

$$\Delta^{\text{V}} \triangleq \{\Delta_i = \begin{bmatrix} -\delta_\nu & \delta_\omega \\ -\delta_\omega & -\delta_\nu \end{bmatrix} : \delta_\nu, \delta_\omega \in \mathcal{R}\}. \quad (2.6)$$

Note that  $\Delta^{\text{I}}$ ,  $\Delta^{\text{II}}$ , and  $\Delta^{\text{III}}$  are standard in the literature, corresponding respectively to complex matrix block uncertainty, repeated complex scalar uncertainty, and repeated real scalar uncertainty.  $\Delta^{\text{IV}}$  is symmetric, real matrix block uncertainty, while  $\Delta^{\text{V}}$  can be used to describe uncertainty in the imaginary and real parts of a structural system represented in real normal form. If the uncertainty is of the form described by  $\Delta^{\text{IV}}$  or  $\Delta^{\text{V}}$ , it is possible to represent the uncertainty by  $\Delta^{\text{III}}$ . However, as discussed in Reference 24 this reformulation leads to increased conservatism and numerical complexity. The ensuing discussion is not restricted to these forms of uncertainty, but they are important special cases and will be used to provide concrete illustrations of the subsequent concepts.

To state the multivariable absolute stability criterion for  $\Delta \in \Delta_\gamma$  we define the sets of Hermitian, frequency-dependent, scaling matrix functions by

$$\begin{aligned} \mathcal{D}_i &\triangleq \{D_i : j\mathcal{R} \cup \infty \rightarrow \mathcal{C}^{k_i \times k_i} : D_i(j\omega) \geq 0, \\ &D_i(j\omega)\Delta_i = \Delta_i D_i(j\omega), \omega \in \mathcal{R}, \Delta_i \in \mathcal{I}_i, i = 1, \dots, p\}, \end{aligned} \quad (2.7)$$

$$\begin{aligned} \mathcal{N}_i &\triangleq \{N_i : j\mathcal{R} \cup \infty \rightarrow \mathcal{C}^{k_i \times k_i} : N_i(j\omega) = N_i^*(j\omega), \\ &N_i(j\omega)\Delta_i = \Delta_i^* N_i(j\omega), \omega \in \mathcal{R}, \Delta_i \in \mathcal{I}_i, i = 1, \dots, p\}. \end{aligned} \quad (2.8)$$

Furthermore, define the sets  $\mathcal{M}_i$  and  $\mathcal{M}$  of multiplier transfer functions by

$$\mathcal{M}_i \triangleq \{M_i(s) = D_i(s) + Q_i(s) : D_i(j\omega) \in \mathcal{D}_i, Q_i(j\omega) = j\omega N_i(j\omega), N_i(j\omega) \in \mathcal{N}_i, i = 1, \dots, p\}, \quad (2.9)$$

$$\mathcal{M} \triangleq \{M(s) \in \mathcal{C}^{m \times m} : M(s) = \text{block-diag}(M_1(s), \dots, M_p(s)), M_i(s) \in \mathcal{M}_i, i = 1, \dots, p\}. \quad (2.10)$$

Note that in (2.9),  $D_i(j\omega) = \text{He } M_i(j\omega) \geq 0$  and  $N_i(j\omega) = -j\text{Sh } M_i(j\omega)$ . Furthermore,  $M(s) \in \mathcal{M}$  satisfies

$$\text{He } M(j\omega) \geq 0, \quad \omega \in \mathcal{R} \cup \infty, \quad (2.11)$$

and is not necessarily stable.

**Theorem 1.** Suppose that  $G(s)[I - \gamma G(s)]^{-1}$  is asymptotically stable. If there exists  $M(s) \in \mathcal{M}$  such that

$$\text{He}[M(j\omega)T_\gamma(j\omega)] > 0, \omega \in \mathcal{R} \cup \infty, \quad (2.12)$$

where

$$T_\gamma(s) \triangleq [I + \gamma G(s)][I - \gamma G(s)]^{-1}, \quad (2.13)$$

then the negative feedback interconnection of  $G(s)$  and  $\Delta$  is asymptotically stable (or, equivalently,  $\det(I + G(j\omega)\Delta) \neq 0, \omega \in \mathcal{R}$ ) for all  $\Delta \in \mathbf{\Delta}_\gamma$ .

**Proof.** A rigorous proof of this result is given in Reference 24. Similar results are presented in References 29-4.  $\triangle$

**Remark 1.** Note that[4]

$$T_\gamma(s) \sim \left[ \begin{array}{c|c} A + \gamma B(I - \gamma D)^{-1}C & \sqrt{2\gamma}B(I - \gamma D)^{-1} \\ \hline \sqrt{2\gamma}(I - \gamma D)^{-1}C & (I + \gamma D)(I - \gamma D)^{-1} \end{array} \right]. \quad (2.14)$$

Using the coprime factorization result presented in Reference 37, it follows that  $M(s)$  can be factored as

$$M(s) = [M_B^*(s)]^{-1}M_A(s), \quad (2.15)$$

where both  $M_A(s)$  and  $M_B(s)$  are asymptotically stable and nonunique. In practice, stable  $M_A(s)$  and  $M_B(s)$  satisfying (2.15) can be computed using the approach pioneered in Reference 31 which is also given in Reference 37. In Reference 35 (2.15) is used to prove the following important corollary to Theorem 1.

**Corollary 1.** Assume  $M(s) \in \mathcal{M}$ , and  $M_A(s)$  and  $M_B(s)$  are asymptotically stable transfer function matrices satisfying (2.15). Then (2.12) in Theorem 1 holds if and only if

$$\text{He}[M_A(j\omega)T_\gamma(j\omega)M_B(j\omega)] > 0, \omega \in \mathcal{R} \cup \infty. \quad (2.16)$$

Corollary 1 allows us to develop robust controllers based on positive real theory. Such a method is developed in Section 3.

We now characterize  $\mathcal{M}_i$  for  $\mathcal{I}_i$  equal to the sets defined by (2.2)-(2.6). These characterizations are restatements of results given in Reference 24. Multiplier sets corresponding to  $\mathbf{\Delta}^I, \mathbf{\Delta}^{II}$ , and  $\mathbf{\Delta}^{III}$  are given in References 29, 35. The set corresponding to  $\mathbf{\Delta}^{II}$  [35] differs from that given here.

The following characterizations are useful in constructing state space realizations of the multipliers. In particular, for  $\mathcal{I}_i = \Delta^{\text{I}}$

$$\mathcal{M}_i = \{m_i(s)I_{k_i} : \text{Re}[m_i(j\omega)] \geq 0, \text{Im}[m_i(j\omega)] = 0, \omega \in \mathcal{R} \cup \infty\}, \quad (2.17)$$

while for  $\mathcal{I}_i = \Delta^{\text{II}}$

$$\mathcal{M}_i = \left\{M_i(s) \in \mathcal{C}^{k_i \times k_i} : \text{He } M_i(j\omega) \geq 0, M(j\omega) = M^*(j\omega), \omega \in \mathcal{R} \cup \infty\right\}. \quad (2.18)$$

Note that if we denote  $\mathcal{M}_i(s) = [m_{jk}(s)]_{(j,k=1,\dots,k_i)}$ ,  $M(j\omega) = M^*(j\omega)$  implies

$$\text{Im}[m_{jj}(j\omega)] = 0, \quad m_{jk}(j\omega) = m_{kj}(-j\omega), \quad j \neq k.$$

Furthermore, for  $\mathcal{I}_i = \Delta^{\text{III}}$

$$\mathcal{M}_i = \left\{M_i(s) \in \mathcal{C}^{k_i \times k_i} : \text{He } M(j\omega) \geq 0, \omega \in \mathcal{R} \cup \infty\right\}, \quad (2.19)$$

while for  $\mathcal{I}_i = \Delta^{\text{IV}}$

$$\mathcal{M}_i = \{m_i(s)I_{k_i} : m_i(s) \in \mathcal{C}, \text{Re}[m_i(j\omega)] \geq 0, \omega \in \mathcal{R} \cup \infty\}. \quad (2.20)$$

Finally, for  $\mathcal{I}_i = \Delta^{\text{V}}$

$$\begin{aligned} \mathcal{M}_i &= \{M_i(s) = D(s) + Q(s) : D(s) = \begin{bmatrix} d_{11}(s) & d_{12}(s) \\ -d_{12}(s) & d_{11}(s) \end{bmatrix}, \\ &Q(s) = \begin{bmatrix} q_{11}(s) & q_{12}(s) \\ q_{12}(s) & -q_{11}(s) \end{bmatrix}, \text{He } D(j\omega) > 0, \\ &\text{Re } Q(j\omega) = 0, \text{Im}[d_{11}(j\omega)] = \text{Re}[d_{12}(j\omega)] = 0, \omega \in \mathcal{R} \cup \infty\}. \end{aligned} \quad (2.21)$$

## 2.1. The Structured Singular Value and Robust Performance

For a multiple block-structured uncertainty set  $\mathcal{I}$ , with possibly repeated scalar elements, complex scalar elements, real blocks, and complex blocks,[24] the structured singular value of a complex matrix  $G(j\omega)$  is defined by

$$\mu(G(j\omega)) \triangleq \left( \inf_{\Delta \in \mathcal{I}} \{\sigma_{\max}(\Delta) : \det(I + G(j\omega)\Delta) = 0\} \right)^{-1},$$

where by convention  $\mu(G(j\omega)) = 0$  if there does not exist  $\Delta \in \mathcal{I}$  such that  $\det(I + G(j\omega)\Delta) = 0$ . The structured singular value nonconservatively characterizes the robust stability of the uncertainty feedback system of Figure 1, as stated by the following theorem.[32, 24]

**Theorem 2.** Suppose  $G(s)$  is asymptotically stable. Then the negative feedback interconnection of  $G(s)$  and  $\Delta$  is asymptotically stable for all  $\Delta \in \mathbf{\Delta}_\gamma$  if and only if

$$\mu(G(j\omega)) < \gamma^{-1}, \omega \in \mathcal{R} \cup \infty. \quad (2.22)$$

**Remark 2.** The parameter  $\gamma_m \triangleq 1/\sup_{\omega \geq 0} \mu(G(j\omega))$  is the multivariable stability margin.[34]

Next define

$$\mu_{\text{abs}}(G(j\omega)) \triangleq \inf\{\gamma > 0 : \text{there exists } M(\cdot) \in \mathcal{M} \text{ such that } \text{He}[M(j\omega)T_\gamma(j\omega)] > 0, \omega \in \mathcal{R} \cup \infty\}. \quad (2.23)$$

Then the following holds.

**Theorem 3.** For  $\omega \in \mathcal{R} \cup \infty$ , let  $G(j\omega)$  be a complex matrix. Then

$$\mu(G(j\omega)) \leq \mu_{\text{abs}}(G(j\omega)).$$

**Proof** A rigorous proof is given in Reference 24. Similar results are stated in References 29, 35.  $\triangle$

The significance of the above theorem is that it allows us to consider both robust stability and robust performance in the same setting.[43] Hence, as was proved for structured singular value analysis with purely complex uncertainty,[32] robust performance can be ensured by appropriate inclusion of a “fictitious” full complex uncertainty block.

## 2.2. Linear Matrix Inequality and Riccati Equation Characterizations of (Generalized) Positive Real Matrix Functions

Theorem 1 characterizes robustness in terms of the strictly generalized positive real condition (2.12) while Corollary 1 relies on the strictly positive real condition (2.16). Hence, to implement the robustness test of Theorem 1 using state space computations requires state space characterizations of strictly generalized positive real and strictly positive real transfer functions.

State space conditions for *strictly* positive real transfer functions are given in Reference 38, but include observability and rank conditions which are difficult to incorporate into numerical schemes. Hence, the lemma below provides state space characterizations of strongly generalized positive real matrices and strongly positive real matrices, special cases respectively of strictly generalized positive real matrices and strictly positive real matrices.

**Lemma 1.** Let  $G(s)$  be square with  $G(s) \sim \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$  where  $(A, B)$  is controllable. Then the following statements are equivalent:

1.  $G(s)$  is strongly generalized positive real.
2. There exist  $\epsilon > 0$  and symmetric  $P$  such that

$$\begin{bmatrix} -A^T P - PA & -PB + C^T \\ -B^T P + C & (D - \epsilon I) + (D - \epsilon I)^T \end{bmatrix} \geq 0.$$

Furthermore, if (1) and (2) hold, then:

4.  $(A, C)$  is observable if and only if  $P$  is nonsingular.
5.  $G(s)$  is strongly positive real if and only if  $A$  is asymptotically stable and  $P \geq 0$ . In this case,  $(A, C)$  is observable if and only if  $P > 0$ .
6. If  $(A, C)$  is observable,  $G(s)$  is strongly positive real if and only if  $D + D^T > 0$  and there exist  $P > 0$  and  $\epsilon > 0$  such that

$$0 = A^T P + PA + (B^T P - C)^T (D + D^T)^{-1} (B^T P - C) + \epsilon I. \quad (2.24)$$

**Proof.** The equivalence of statements 1 and 2 is shown in Reference 29. Statements 3 and 4 are proved in Reference 2 and statement 5 is proved in Reference 19.  $\triangle$

Lemma 1 shows that strongly generalized positive reality and generalized positive reality is equivalent to the existence of a certain solution to an LMI. Furthermore, the latter condition (under the assumption that  $(A, B, C, D)$  is a minimal state space realization) is equivalent to the existence of a certain solution to a Riccati equation. This Riccati equation is subsequently used to develop probability-one homotopy maps for robust controller synthesis with fixed-structure multipliers.

### 2.3. Bilinear Matrix Inequality Approaches to Robust Controller Synthesis

Both References 29, 4 describe ways of using the LMI's corresponding to generalized positive real tests to develop robustness tests expressed in terms of the existence of a solution to an LMI. The key is to represent the multiplier  $M(s)$  in the strictly generalized positive real test (2.12) of Theorem 1 in such a way that its parameters appear linearly in the corresponding state space test of Lemma 1. When the LMI robustness *analysis* results are generalized to fixed-architecture, robust control *synthesis*, a BMI results.

A straightforward way of achieving the desired representation of  $M(s)$  is described in Reference 4 and is based on earlier ideas given in Reference 35. This approach requires expressing the multiplier  $M(s)$  in terms of a basis expansion. In particular,

$$M(s) = \sum_{j=1}^r \beta_j \bar{M}^{(j)}(s), \quad (2.25)$$

where  $\beta_j \geq 0$  and  $\bar{M}^{(j)}(s) \in \mathcal{M}$ ,  $j = 1, \dots, r$ . A substantial weakness of this approach is the difficulty of choosing the basis  $\{\bar{M}^{(1)}(s), \dots, \bar{M}^{(r)}(s)\}$ . Specifically, for a given multiplier  $M(s) \in \mathcal{M}$  and a given basis, the approach in Reference 4 does not guarantee that there exists nonnegative  $\beta_j$  such that  $M(s)$  is given by (2.25).

The approach proposed in Reference 29 does not suffer from these weaknesses. It is based on the following result.

**Lemma 2.** [35] Equation (2.12) is satisfied for some transfer matrix  $M^{(0)}(s) \in \mathcal{M}$  if and only if there exist a real polynomial matrix  $M(s) \in \mathcal{M}$  for which (2.12) holds. Furthermore, if  $M^{(0)}(s)$  is factored as  $M^{(0)}(s) = \frac{1}{d^{(0)}(s)} N^{(0)}(s)$  where  $N^{(0)}(s)$  is a real polynomial matrix and  $d^{(0)}(s)$  is a scalar-valued real polynomial, then the degree of  $M(s)$  need not be greater than the sum of degrees of  $N^{(0)}(s)$  and  $d^{(0)}(s)$ , and in fact one can choose  $M(s) = d^{(0)}(-s)N^{(0)}(s)$ . In addition, if  $M(s)$  is of order  $2n$ , then for all  $n^{\text{th}}$ -order real polynomials  $d(s)$  having no zeros on the  $j\omega$  axis,  $\tilde{M}(s) = \frac{M(s)}{d(-s)d(s)} \in \mathcal{M}$  and satisfies (2.12).

**Remark 3.** Lemma 2 allows one to restrict the multiplier search to  $2n^{\text{th}}$  order, real polynomial matrices  $M(s)$ . To obtain state space realizable transfer functions we can consider

$$\tilde{M}(s) = \frac{M(s)}{d(s)d(-s)}, \quad (2.26)$$

where  $d(s)$  is an arbitrary  $n^{\text{th}}$ -order polynomial having no zeros on the  $j\omega$ -axis.

**Remark 4.** Note that since the zeros of  $d(-s)$  are the mirror images of the zeros of  $d(s)$  about the imaginary axis,  $\tilde{M}(s)$  given by (2.26) is *always* an unstable multiplier.

This latter approach is powerful but as eluded to in Remark 4 always produces an unstable multiplier. When extended to fixed architecture, robust control design, this approach results in a BMI with very high dimension since it requires the introduction of a Lyapunov inequality of dimension equal to that of the closed-loop system to ensure closed-loop stability. We hence begin the development of an alternative scheme without these limitations.

### 3. A Nonlinear Matrix Inequality Approach to Robust Controller Synthesis with Fixed-Structure Multipliers

We now give exclusive attention to robustness tests that are expressed in terms of positive real conditions, as opposed to generalized positive real conditions. Hence, we will consider robustness tests corresponding to Corollary 1.

We focus on the uncertainty structures corresponding to the sets  $\Delta^I$  and  $\Delta^{III}$ , defined respectively by (2.2) and (2.4). Hence, we develop fixed-structure multiplier tests for the complex, block-structured uncertainty considered by classical complex structured singular value analysis [11, 32] and the real, diagonal uncertainty considered by classical real structured singular value analysis.[12, 43] Although not detailed here, the uncertainty structures corresponding to the sets  $\Delta^{II}$ ,  $\Delta^{IV}$ , and  $\Delta^V$ , may also be considered in the framework of this section. In addition, the results may be extended in a straightforward manner to mixed uncertainty sets.

Below we develop constructive characterizations of multiplier factors  $M_A(s)$  and  $M_B(s)$  corresponding to complex, block-structured uncertainty and real, diagonal uncertainty. Robustness tests are then formulated as NMI's and are subsequently extended to NMI's for fixed-structure, robust control.

#### 3.1. Complex, Block-Structured Uncertainty

From (2.10) and (2.17) it follows that a multiplier corresponding to complex block-diagonal uncertainty is given by

$$M(s) = \text{block-diag} \left( m_1(s)I_{k_1}, \dots, m_p(s)I_{k_p} \right), \quad (3.1)$$

$$\text{Re } m_i(j\omega) > 0, \quad \omega \in \mathcal{R} \cup \infty, \quad (3.2)$$

$$\text{Im } m_i(j\omega) = 0, \quad \omega \in \mathcal{R} \cup \infty. \quad (3.3)$$

Note that we have replaced the weak inequality in (2.17) with a strict inequality in (3.2).

**Lemma 3.** For uncertainty structures corresponding to (2.2) there exists  $\tilde{M}(s)$  with no zeros or poles on the imaginary axis and satisfying the compatibility conditions (3.1)-(3.3) and the robustness test (2.12), if and only if there exists  $M(s)$  satisfying (2.12) and (3.1) such that

$$M(s) = \bar{M}(-s)\bar{M}(s), \quad (3.4)$$

where  $\bar{M}(s)$  has no poles or zeros in the closed right half plane.

**Proof.** The proof is given in [10].  $\triangle$

**Remark 5.** Equation (3.4) corresponds to the stable coprime factorization of  $M(s)$  given by (2.15) with  $M_A(s) = \bar{M}(s)$  and  $M_B(s) = \bar{M}^{-1}(s)$ . Furthermore, if  $\bar{M}(s)$  is strictly proper then  $M(s)$  given by (3.4) is strongly positive real and  $\bar{M}(s)$  and  $\bar{M}^{-1}(s)$  both have state space realizations.

**Remark 6.**  $\bar{M}(s)$  is precisely an asymptotically stable, minimum phase transfer function representation of the classical  $D$ -scales from complex structured singular value analysis.[11, 32]

From Remark 5, it follows that if we denote the state space realization of strictly proper  $M_A(s)$  in (2.15) by

$$M_A(s) \sim \left[ \begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array} \right], \quad (3.5)$$

then, we can simply choose  $M_B(s) = M_A^{-1}(s)$  and hence using a standard state space realization inversion formula [30]

$$M_B(s) \sim \left[ \begin{array}{c|c} \bar{A} - \bar{B}\bar{D}^{-1}\bar{C} & \bar{B}\bar{D}^{-1} \\ \hline -\bar{D}^{-1}\bar{C} & \bar{D}^{-1} \end{array} \right]. \quad (3.6)$$

Details on how to choose  $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$  to enforce the block-diagonal structure of  $M_A(s)$  are given in Reference 21.

Now, if we let  $T_\gamma(s)$  have the state space realization  $(A_\gamma, B_\gamma, C_\gamma, D_\gamma)$  as in (2.14), then, referring to (2.16),

$$M_A(s)T_\gamma(s)M_B(s) \sim \left[ \begin{array}{c|c} \tilde{A}_\gamma & \tilde{B}_\gamma \\ \hline \tilde{C}_\gamma & \tilde{D}_\gamma \end{array} \right],$$

where

$$\tilde{A}_\gamma = \begin{bmatrix} \bar{A} - \bar{B}\bar{D}^{-1}\bar{C} & 0 & 0 \\ -B_\gamma\bar{D}^{-1}\bar{C} & A_\gamma & 0 \\ -\bar{B}D_\gamma\bar{D}^{-1}\bar{C} & \bar{B}B_\gamma & \bar{A} \end{bmatrix}, \quad \tilde{B}_\gamma = \begin{bmatrix} \bar{B}\bar{D}^{-1} \\ B_\gamma\bar{D}^{-1} \\ \bar{B}D_\gamma\bar{D}^{-1} \end{bmatrix}, \quad (3.7)$$

$$\tilde{C}_\gamma = \begin{bmatrix} -\bar{D}D_\gamma\bar{D}^{-1}\bar{C} & \bar{D}D_\gamma & \bar{C} \end{bmatrix}, \quad \tilde{D}_\gamma = \bar{D}D_\gamma\bar{D}^{-1}. \quad (3.8)$$

The next theorem which considers complex, block-structured uncertainty, follows immediately from Corollary 1 and Lemma 1.

**Theorem 4.** Let  $\mathcal{I}_i = \mathbf{\Delta}^I$ ,  $i = 1, \dots, p$ , and suppose  $G(s)$  is asymptotically stable. If there exist  $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ ,  $P > 0$ , and  $\epsilon > 0$  such that

$$\begin{bmatrix} \tilde{A}_\gamma^T P + P\tilde{A}_\gamma & -P\tilde{B}_\gamma + \tilde{C}_\gamma^T & 0 \\ -\tilde{B}_\gamma^T P + \tilde{C}_\gamma & (\tilde{D}_\gamma - \epsilon I) + (\tilde{D}_\gamma - \epsilon I)^T & 0 \\ 0 & 0 & P \end{bmatrix} > 0$$

where  $\tilde{A}_\gamma, \tilde{B}_\gamma, \tilde{C}_\gamma, \tilde{D}_\gamma$  are given by (3.7) and (3.8), then the negative feedback interconnection of  $G(s)$  and  $\Delta$  is asymptotically stable for all  $\Delta \in \mathbf{\Delta}_\gamma$ .

### 3.2. Real, Diagonal Uncertainty

Recall from (2.10) and (2.19) that a multiplier corresponding to real, diagonal uncertainty (with possible repeated elements) is given by

$$M(s) = \text{block-diag}(M_1(s), \dots, M_p(s)),$$

where

$$\text{He}[M(j\omega)] \geq 0, \quad \omega \in \mathcal{R} \cup \infty. \quad (3.9)$$

Now, if we let  $M(s) = M_B^*(s)^{-1}M_A(s)$  as in (2.15), then it follows from Corollary 1 that if we consider only strict inequality in (3.9), then we can replace (3.9) with

$$\text{He}[M_A(j\omega)M_B(j\omega)] > 0, \quad \omega \in \mathcal{R} \cup \infty. \quad (3.10)$$

Next let the state space realizations of  $M_A(s)$  and  $M_B(s)$  be denoted, respectively, by

$$M_A(s) \sim \left[ \begin{array}{c|c} A_A & B_A \\ \hline C_A & D_A \end{array} \right], \quad M_B(s) \sim \left[ \begin{array}{c|c} A_B & B_B \\ \hline C_B & D_B \end{array} \right].$$

If we let  $T_\gamma(s)$  have the state space realization  $(A_\gamma, B_\gamma, C_\gamma, D_\gamma)$  as in (2.14), then referring to (2.16)

$$M_A(s)T_\gamma(s)M_B(s) \sim \left[ \begin{array}{c|c} \tilde{A}_{\gamma,1} & \tilde{B}_{\gamma,1} \\ \hline \tilde{C}_{\gamma,1} & \tilde{D}_{\gamma,1} \end{array} \right],$$

where

$$\begin{aligned} \tilde{A}_{\gamma,1} &= \begin{bmatrix} A_B & 0 & 0 \\ B_\gamma C_B & A_\gamma & 0 \\ B_A D_\gamma C_B & B_A C_\gamma & A_A \end{bmatrix}, & \tilde{B}_{\gamma,1} &= \begin{bmatrix} B_B \\ B_\gamma D_B \\ B_A D_\gamma D_B \end{bmatrix}, \\ \tilde{C}_{\gamma,1} &= \begin{bmatrix} D_A D_\gamma D_B & D_A C_\gamma & C_A \end{bmatrix}, & \tilde{D}_{\gamma,1} &= D_A D_\gamma D_B. \end{aligned}$$

Similarly, referring to (3.10),

$$M_A(s)M_B(s) \sim \left[ \begin{array}{c|c} \tilde{A}_{\gamma,2} & \tilde{B}_{\gamma,2} \\ \hline \tilde{C}_{\gamma,2} & \tilde{D}_{\gamma,2} \end{array} \right],$$

where

$$\begin{aligned} \tilde{A}_{\gamma,2} &= \begin{bmatrix} A_B & 0 \\ B_A C_B & A_A \end{bmatrix}, & \tilde{B}_{\gamma,2} &= \begin{bmatrix} B_B \\ B_A D_B \end{bmatrix}, \\ \tilde{C}_{\gamma,2} &= \begin{bmatrix} D_A C_B & C_A \end{bmatrix}, & \tilde{D}_{\gamma,2} &= D_A D_B. \end{aligned}$$

Now, let

$$\tilde{A}_\gamma = \text{block-diag}\{\tilde{A}_{\gamma,1}, \tilde{A}_{\gamma,2}\}, \quad \tilde{B}_\gamma = \text{block-diag}\{\tilde{B}_{\gamma,1}, \tilde{B}_{\gamma,2}\}, \quad (3.11)$$

$$\tilde{C}_\gamma = \text{block-diag}\{\tilde{C}_{\gamma,1}, \tilde{C}_{\gamma,2}\}, \quad \tilde{D}_\gamma = \text{block-diag}\{\tilde{D}_{\gamma,1}, \tilde{D}_{\gamma,2}\}. \quad (3.12)$$

The next theorem, follows immediately from Corollary 1 and Lemma 1.

**Theorem 5.** Let  $\mathcal{I}_i = \mathbf{\Delta}^{\text{III}}$ ,  $i = 1, \dots, p$ , and suppose  $G(s)$  is asymptotically stable. If there exist  $(A_A, B_A, C_A, D_A)$ ,  $(A_B, B_B, C_B, D_B)$ ,  $P > 0$ , and  $\epsilon > 0$  such that  $\tilde{D}_\gamma + \tilde{D}_\gamma^T > 0$  and

$$\begin{bmatrix} \tilde{A}_\gamma^T P + P \tilde{A}_\gamma & -P \tilde{B}_\gamma + \tilde{C}_\gamma^T & 0 \\ -\tilde{B}_\gamma^T P + \tilde{C}_\gamma & (\tilde{D}_\gamma - \epsilon I) + (\tilde{D}_\gamma - \epsilon I)^T & 0 \\ 0 & 0 & P \end{bmatrix} > 0$$

where  $\tilde{A}_\gamma, \tilde{B}_\gamma, \tilde{C}_\gamma, \tilde{D}_\gamma$  are given by (3.11) and (3.12), then the negative feedback interconnection of  $G(s)$  and  $\Delta$  is asymptotically stable for all  $\Delta \in \mathbf{\Delta}_\gamma$ .

### 3.3. Problem Formulation

Both Theorems 4 and 5 provide robustness tests in terms of the following feasibility problem.

**Nonlinear Matrix Inequality for Robust Controller Synthesis (NMIRCS)** . Find  $\theta \in \mathcal{R}^q$ ,  $\epsilon > 0$ , and  $P \in \mathcal{R}^{r \times r}$  such that

$$\begin{bmatrix} \tilde{A}_\gamma^T(\theta)P + P\tilde{A}_\gamma(\theta) & -P\tilde{B}_\gamma(\theta) + \tilde{C}_\gamma^T(\theta) & 0 \\ -\tilde{B}_\gamma^T(\theta)P + \tilde{C}_\gamma(\theta) & (\tilde{D}_\gamma(\theta) - \epsilon I) + (\tilde{D}_\gamma(\theta) - \epsilon I)^T & 0 \\ 0 & 0 & P \end{bmatrix} > 0 \quad (3.13)$$

where the dimension  $q$  is determined by the multiplier and  $r$  is determined by both the multiplier and the nominal plant size. In Theorems 4 and 5,  $\theta$  corresponds to the free parameters of the matrices providing a state-space representation of the multiplier factors  $M_A(s)$  and  $M_B(s)$ . For example, considering Theorem 4, if all of the elements of the matrices  $\bar{A}, \bar{B}, \bar{C}$ , and  $\bar{D}$  in (3.5) and (3.6) are free, then  $\theta$  is defined by  $\theta = \left( \text{vec}^T(\bar{A}), \text{vec}^T(\bar{B}), \text{vec}^T(\bar{C}), \text{vec}^T(\bar{D}) \right)^T$ .

If we are considering control design for a plant  $(A_p, B_p, C_p, D_p)$  under a feedback controller  $(A_c, B_c, C_c)$ , then, assuming negative feedback,  $A$  in (2.14) is given by

$$A = \begin{bmatrix} A_p & -B_p C_c \\ B_c C_p & A_c - B_c D_p C_c \end{bmatrix}. \quad (3.14)$$

Hence, in Theorems 4 and 5  $\tilde{A}_\gamma$  is linear in the controller matrices. The controller matrices essentially provide extra degrees of freedom to satisfy the Riccati equation constraint (3.13). To illustrate,

if all of the elements of the matrices  $\bar{A}$ ,  $\bar{B}$ ,  $\bar{C}$ , and  $\bar{D}$  are free, and all of the matrices  $A_c$ ,  $B_c$ ,  $C_c$  are also free then  $\theta$  is defined by  $\theta = (\text{vec}^T(\bar{A}), \text{vec}^T(\bar{B}), \text{vec}^T(\bar{C}), \text{vec}^T(\bar{D}), \text{vec}^T(A_c), \text{vec}^T(B_c), \text{vec}^T(C_c))^T$ . Note that if  $\tilde{A}_\gamma(\theta)$ ,  $\tilde{B}_\gamma(\theta)$ ,  $\tilde{C}_\gamma(\theta)$ , and  $\tilde{D}_\gamma(\theta)$  are affine functions of  $\theta$ , then the NMI (3.13) is a BMI.

## 4. A Continuation Algorithm for Nonlinear Matrix Inequality Feasibility Problems

Let  $G(\cdot)$  be a nonlinear function mapping  $\mathcal{R}^n \times \mathcal{R} \longrightarrow \mathcal{R}^{p \times p}$  and define

$$F_{\gamma^*}(x) \triangleq G(x, \gamma)|_{\gamma=\gamma^*} \quad (4.1)$$

Given  $\gamma_f$ , we desire to find  $x$  such that

$$F_{\gamma_f}(x) > 0. \quad (4.2)$$

It is assumed that given  $x_0$  there exists  $\gamma_0$  such that

$$F_{\gamma_0}(x_0) > 0. \quad (4.3)$$

Let

$$\gamma(\lambda) = (1 - \lambda)\gamma_0 + \lambda\gamma_f \quad (4.4)$$

and define

$$H(\lambda, x) \triangleq G(x, \gamma(\lambda)). \quad (4.5)$$

Consider the NMI

$$H(\lambda, x) > 0. \quad (4.6)$$

Note that

$$H(0, x) = F_{\gamma_0}(x) \quad (4.7)$$

and hence at  $\lambda = 0$ , (4.6) has a known solution  $x_0$ , i.e.,  $H(0, x_0) > 0$ . Also,

$$H(1, x) = F_{\gamma_f}(x) \quad (4.8)$$

and hence at  $\lambda = 1$  (4.6) becomes the desired NMI (4.2).

To enable path following for some  $c \in \mathcal{R}^n$  we introduce the linear cost functional

$$J(x) = c^T x. \quad (4.9)$$

It is desired to solve the optimization problem

$$\min_x J(x) \text{ subject to } F_{\gamma_f} > 0. \quad (4.10)$$

To accomplish this we consider the curve defined by

$$\min_x J(x) \text{ subject to } H(\lambda, x) > 0, \lambda \in [0, 1). \quad (4.11)$$

Before presenting a continuation algorithm to follow the path defined by (4.11), define the linear function

$$L_{\gamma^*, x_0}(x) = \sum_{i=1}^n \frac{\partial F_{\gamma^*}}{\partial x_i} \Big|_{x=x_0} x_i.$$

The affine function

$$\bar{F}_{\gamma^*}(x) = \left[ F_{\gamma^*}(x_0) - \sum_{i=1}^n \frac{\partial F}{\partial x_i} \Big|_{x=x_0} x_{0,i} \right] + L_{\gamma^*, x_0}(x)$$

denotes the linearization of  $F_{\gamma^*}(x)$  about  $x_0$  while

$$\bar{H}(\lambda) = H(\lambda_0, x(\lambda_0)) + L_{\gamma_0, x(\lambda_0)}(x'(\lambda_0)(\lambda - \lambda_0))$$

where

$$x'(\lambda) \triangleq \frac{\partial x}{\partial \lambda}$$

denotes the linearization of  $H(\lambda, x(\lambda))$  about  $\lambda_0$ .

#### 4.1. Continuation Algorithm

1. Set  $\lambda = 0$ ,  $\gamma = \gamma(0)$ ,  $x^{(0)} = x_0$ .
2. For  $k = 0, 1, 2, \dots$  until convergence compute  $x^{(k+1)}$  by solving the affine matrix inequality optimization problem

$$\min_{x^{(k+1)}} c^T x^{(k+1)}$$

subject to

$$\left[ F_{\gamma}(x^{(k)}) - \sum_{i=1}^n \frac{\partial F}{\partial x_i} \Big|_{x=x^{(k)}} x_i^{(k)} \right] + L_{\gamma, x^{(k)}}(x^{(k+1)}) > 0.$$

Let  $x(\lambda) = \lim_{k \rightarrow \infty} x^{(k)}$ .

3. If  $\lambda = 1$ , let  $x = x(1)$  and stop.

4. Compute the tangent vector  $x'(\lambda)$  by solving the affine matrix inequality optimization problem

$$\min_{x'(\lambda)} c^T x'(\lambda)$$

subject to

$$H(\lambda, x(\lambda)) + L_{\gamma, x(\lambda)}(x'(\lambda)) > 0.$$

5. For  $\lambda \leftarrow \lambda + \Delta\lambda \leq 1$ , predict  $x(\lambda + \Delta\lambda)$  using polynomial interpolation. Let  $x_1$  denote this prediction.
6. Set  $\gamma = \gamma(\lambda + \Delta\lambda)$ ,  $x^{(0)} = x_1$  and go to step (2).

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