A Delta Operator Approach to Discrete-Time H_∞ Control

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Emmanuel G. Collins, Jr. and Tinglun Song
Department of Mechanical Engineering
Florida A&M University – Florida State University
2525 Pottsdamer St. Tallahassee, FL 32310-6046
ecollins@eng.fsu.edu, song@eng.fsu.edu

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Abstract

To increase the performance of digitally implemented control laws, it is desirable to directly synthesize the digital control law. Thus, this paper considers the direct synthesis of discrete-time, H_∞ control laws. Previously two approaches have been proposed to this problem. The most common method uses the bilinear transformation to convert the discrete-time problem into an equivalent continuous-time problem, allowing the controller to be synthesized with software developed for continous-time systems. This method is often effective but requires unnecessary steps and is not applicable in all circumstances. The second approach is to synthesize the discrete-time, H_∞ control law by solving discrete-time, H_∞ Riccati equations. Due to numerical ill-conditioning, this approach can fail at sufficiently high sample rates. This numerical ill-conditioning can be eliminated by representing the discrete-time system and H_∞ controller in the delta domain. Hence, the H_∞ design equations for direct synthesis in the delta domain are developed. This paper also unifies the continuous-time and discrete-time H_∞ design theories by showing that when the sample period $\Delta = 0$, the delta-domain H_∞ design equations reduce to the known continuous-time H_∞ design equations.

Keywords: H_∞ control, discrete-time, delta operator

Running Title: Delta-domain H_∞ Control

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1. Introduction

Much of the available literature on H_∞ control has focused on the treatment of continuous-time systems (Doyle and Glover 1989, Gahinet 1996, Green and Limebeer 1995, Iwasaki and Skelton 1994, Peterson *et al.* 1991, Stoovogel 1992b, Zhou *et al.* 1995). In contrast, the discrete-time control problem (Green and Limebeer 1995, Iglesias and Glover 1991, Limebeer and Green 1989, Stoovogel 1992a,b) has not received much attention although discrete-time controllers are commonly used in many applications.

There are two primary methods used to develop full-order discrete-time controllers. One approach is to apply the bilinear transformation $z = \frac{1+s}{1-s}$ which maps discrete-time systems to continuous-time systems isometrically. After this transformation, continuous-time H_∞ controller synthesis can be used to obtain a continuous-time controller which can then be mapped back to the discrete-time domain by using an inverse transformation $s = \frac{z-1}{z+1}$. The second approach is to use the discrete-time controller synthesis methods developed in the discrete-time domain directly.

There are some disadvantages associated with the commonly used bilinear transform approach. First, it maps systems with a pole at 1 into improper systems and hence, in this case is not valid (Stoovogel 1992b). It also introduces unnecessary complexity since it is possible to develop direct design methods that avoid the transformation steps. In addition, the bilinear transformation cannot be applied to time-varying or finite-horizon problems.

In recent years, discrete-time H_∞ design equations were developed which can be solved to obtain discrete-time controllers directly (Green and Limebeer 1995, Iglesias and Glover 1991, Limebeer and Green 1989, Stoovogel 1992a,b). Each of these references used the standard forward-shift (or z-domain) representation of discrete-time systems in the development of discrete-time H_∞ controller synthesis algorithms. Unfortunately, algorthms to solve these discrete-time equations experience numerical ill conditioning when the sample period is sufficiently small.

Hence, in this paper, the delta representation (Feuer and Goodwin 1996, Goodwin and Middleton 1992, Middleton and Goodwin 1990) is used in place of the usual forward-shift (or z-domain) representation. As shown in (Feuer and Goodwin 1996, Goodwin and Middleton 1992, Middleton and Goodwin 1990) and in this paper, the delta representation of a discrete-time system converges to the underlying continuous systems when the sampling period tends to zero, leading to robust numerical conditioning at small sample periods.

Based on the delta representation, a set of discrete-time H_∞ design equations are derived. The results given in this paper are analogous to those results based on the usual forward-shift representation provided in (Green and Limebeer 1995, Limebeer and Green 1989, Stoovogel 1992a,b). It is shown that the solution of the delta H_∞ controller requires the solution of two delta-domain Riccati equations and converges to the corresponding continuous-time results as the sampling period decreases. This corresponds to previous results developed for H_2 optimal control (Goodwin and Middleton 1992, Middleton and Goodwin 1990). The delta design equations also allow the same software to be used for both discrete-time and continuous-time H_∞ controller synthesis. In addition, it has been demonstrated (Feuer and Goodwin 1996, Goodwin and Middleton 1992, Middleton and Goodwin 1990) that implementation of delta controllers is significantly less sensitive to round-off errors, round-off noise and limit cycles than implementation of the more standard z-domain controllers at high sampling rate.

The paper is organized as follows. Section 2 gives a brief introduction to the delta representation of discrete-time systems. Section 3 presents the discrete-time H_∞ control design equations using the forward-shift representation. Section 4 uses the results of section 3 to derive the discrete-time H_∞ control design equations for the delta representation of a discrete-time system. In Section 5, explicit connections between the continuous-time and discrete-time H_∞ design theory have been drawn. Section 6 provides an illustrative example and section 7 concludes the paper.

2. Discrete-Time Representation Using the Delta-Operator

Consider a linear continuous-time system

$$\dot{x} = Ax + B_1 w + B_2 u, \tag{1}$$

$$z_p = C_1 x + D_{11} w + D_{12} u, \tag{2}$$

$$y = C_2 x + D_{21} w + D_{22} u \tag{3}$$

where $x \in \mathcal{C}^n$ is the state, $y \in \mathcal{R}^q$ is the measurement, $u \in \mathcal{R}^m$ is the control input, $z \in \mathcal{R}^p$ is the performance output, and $w \in \mathcal{R}^l$ is the exogenous input.

Let Δ be the sample period and q the standard forward-shift operator, i.e., $qx(t) = x(t + \Delta)$. Then, if $u(t)$ is processed via a zero-order-hold, the corresponding discrete-time model in forward-shift form is

$$qx = A_q x + B_{1q} w + B_{2q} u, \tag{4}$$

$$z_p = C_1 x + D_{11} w + D_{12} u, \tag{5}$$

$$y = C_2 x + D_{21} w + D_{22} u \tag{6}$$

where

$$A_q = e^{A\Delta}, \qquad B_{1q} = \int_0^{\Delta} e^{A(\Delta-\tau)} B_1 d\tau, \qquad B_{2q} = \int_0^{\Delta} e^{A(\Delta-\tau)} B_2 d\tau. \tag{7}$$

A problem with this representation is that irregardless of the underlying continuous-time system

$$\lim_{\Delta\to 0} A_q = I, \quad \lim_{\Delta\to 0} B_{1q} = 0, \quad \lim_{\Delta\to 0} B_{1q} = 0. \tag{8}$$

As a result, for fast sampling the standard forward-shift representation of a discrete-time system becomes extremely sensitive to roundoff errors.

An alternative discrete-time representation is based on the delta-operator (Feuer and Goodwin 1996, Goodwin and Middleton 1992, Middleton and Goodwin 1990) which is defined as

$$\delta(\Delta) \triangleq \begin{cases} \frac{d}{dt}(\cdot) & : \ \Delta = 0, \\ (q-1)/\Delta & : \ \Delta \neq 0. \end{cases} \tag{9}$$

Using the delta-operator, a discrete-time representation of the previous model is given by

$$\delta x = A_\delta x + B_{1\delta} w + B_{2\delta} u, \tag{10}$$

$$z_p = C_1 x + D_{11} w + D_{12} u, \tag{11}$$

$$y = C_2 x + D_{21} w + D_{22} u \tag{12}$$

where

$$\begin{aligned} A_\delta &= (A_q - I)/\Delta = \Psi(A,\Delta)A, \\ B_{1\delta} &= B_{1q}/\Delta = \Psi(A,\Delta)B_1, \\ B_{2\delta} &= B_{2q}/\Delta = \Psi(A,\Delta)B_2, \\ \Psi(A,\Delta) &= I + \frac{A\Delta}{2!} + \frac{A^2\Delta^2}{3!} + \ldots \end{aligned}$$

and

$$\lim_{\Delta\to 0} A_\delta = A, \qquad \lim_{\Delta\to 0} B_{1\delta} = B_1, \qquad \lim_{\Delta\to 0} B_{2\delta} = B_2. \tag{13}$$

Equation (13) implies that the delta representation converges to the continuous-time representation as the sampling rate increases. Hence the use of the delta operator not only leads to better numerical conditioning of numerical algorithms, but also ensures that the discrete-time results converge to the continuous results as $\Delta \to 0$. This latter feature is useful in unifying discrete-time and continuous-time control theory and software.

Before presenting design equations for H_∞ control, we will define some notations. The variable γ is the delta-operator variable and is analogous to the Laplace variable s for continuous-time systems and the Z transform variable z for z-domain, discrete-time systems. Let $\sigma_{max}(\cdot)$ denote the maximum singular value. The H_∞ norm of the asymptotically stable, continuous-time transfer function $G(s)$ is given by

$$\|G(s)\|_\infty = \sup_{\omega\in\mathcal{R}} \sigma_{max}(G(j\omega)). \tag{14}$$

The H_∞ norm of the asymptotically stable, z-domain transfer function $G(z)$ is given by

$$\|G(z)\|_\infty = \sup_{\theta\in[0,2\pi]} \sigma_{max}(G(e^{j\theta})) \tag{15}$$

while the H_∞ norm of the asymptotically stable, delta-domain transfer function $G(\gamma)$ is given by

$$\|G(\gamma)\|_\infty = \sup_{\omega\in\mathcal{R}} \sigma_{max}(G(\frac{e^{j\omega\Delta}-1}{\Delta})). \tag{16}$$

Note that $G(\gamma)$ is asymptotically stable if its poles lie in the open circle with radius $\frac{1}{\Delta}$ and center at $-\frac{1}{\Delta}$ (Goodwin and Middleton 1992, Middleton and Goodwin 1990).

3. Discrete-time H_∞ Control Using the Forward-Shift Representation

For the discrete-time plant (4)-(6) in forward-shift form, the H_∞ design problem seeks a causal, linear, time-invariant, finite-dimensional controller $K(z)$ such that the closed-loop system is internally stable and for some $\gamma > 0$ satisfies

$$\|G_{z_p w}(P(z), K(z))\|_\infty < \gamma. \tag{17}$$

Here $P(z)$ is the transfer function representation of (4)-(6) and $G_{z_p w}(P(z), K(z))$ is given by the following lower linear fractional transformation (LFT)

$$G_{z_p w}(P(z), K(z)) = P_{11}(z) + P_{12}(z)K(z)(I + P_{22}(z)K(z))^{-1}P_{21}(z). \tag{18}$$

The following theorem provides necessary and sufficient conditions for the existence of such a discrete-time dynamic controller and characterizes one such controller.

Theorem 1. (Green and Limebeer 1995, Stoovogel 1992b). Consider the system given by equations (4)-(6) and assume the following assumptions hold:

1. (A_q, B_{2q}, C_2) is stabilizable and detectable,

2. $D_{12}^T D_{12} > 0$ and $D_{21} D_{21}^T > 0$,

3. $\operatorname{rank}\begin{bmatrix} A_q - e^{j\theta}I & B_{2q} \\ C_1 & D_{12} \end{bmatrix} = n+m,\ \theta \in (-\pi, \pi],$

4. $\operatorname{rank}\begin{bmatrix} A_q - e^{j\theta}I & B_{1q} \\ C_2 & D_{21} \end{bmatrix} = n+q,\ \theta \in (-\pi, \pi].$

Then a causal, linear, finite-dimensional controller exists which guarantees closed-loop internal stability and satisfies the H_∞ norm requirement (17) if and only if the following two conditions hold.

1. The Riccati equation

$$X_q = \bar{C}^T \bar{J} \bar{C} + A_q^T X_q A_q - \bar{L}_q^T \bar{R}_q^{-1} \bar{L}_q \tag{19}$$

where

$$\bar{R}_q = \bar{D}^T \bar{J} \bar{D} + B_q^T X_q B_q,$$
$$\bar{L}_q = \bar{D}^T \bar{J} \bar{C} + B_q^T X_q A_q$$

has a solution such that $A_q - B_q \bar{R}_q^{-1} \bar{L}_q$ is asymptotically stable and

$$X_q \geq 0, \tag{20}$$
$$\bar{R}_{q1} - \bar{R}_{q2}^T \bar{R}_{q3}^{-1} \bar{R}_{q2} < 0 \tag{21}$$

where

$$\bar{C} = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}, \qquad \bar{D} = \begin{bmatrix} D_{11} & D_{12} \\ I_l & 0 \end{bmatrix}, \qquad \bar{J} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_l \end{bmatrix}$$

and $\bar{R}_{q1}, \bar{R}_{q2}$, and $\bar{R}_{q3}$ are given by compatible partition of $\bar{R}_q$.

2. The Riccati equation

$$Z_q = \hat{B}_q \hat{J} \hat{B}_q^T + \hat{A}_q Z_q \hat{A}_q^T - \hat{L}_q \hat{R}_q^{-1} \hat{L}_q^T \tag{22}$$

where

$$\begin{aligned} \hat{R}_q &= \hat{D} \hat{J} \hat{D}^T + \hat{C} Z_q \hat{C}^T, \\ \hat{L}_q &= \hat{B}_q \hat{J} \hat{D}^T + \hat{A}_q Z_q \hat{C}^T, \\ \hat{J} &= \begin{bmatrix} I_l & 0 \\ 0 & -\gamma^2 I_m \end{bmatrix}, \end{aligned}$$

$$\left[\begin{array}{c|c} \hat{A}_q & \hat{B}_q \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{c|cc} A_q + B_{1q} R_{qd}^{-1} L_{qd} & B_{1q} V_{q21}^{-1} & 0 \\ \hline V_{q12} \bar{R}_{q3}^{-1} (\bar{L}_{q2} - \bar{R}_{q2} R_{qd}^{-1} L_{qd}) & V_{q12} \bar{R}_{q3}^{-1} \bar{R}_{2q} V_{q21}^{-1} & I \\ C_2 - D_{21} R_{qd}^{-1} L_{qd} & D_{21} V_{q21}^{-1} & 0 \end{array}\right], \tag{23}$$

$$
\begin{aligned}
R_{qd} &= \bar{R}_{q1} - \bar{R}_{q2}^T \bar{R}_{q3}^{-1} \bar{R}_{q2}, \\
L_{qd} &= \bar{L}_{q1} - \bar{R}_{q2}^T \bar{R}_{q3}^{-1} \bar{L}_{q2}, \\
V_{q12}^T V_{q12} &= \bar{R}_{q3}, \\
V_{q21}^T V_{q21} &= -\gamma^{-2} (\bar{R}_{q1} - \bar{R}_{q2}^T \bar{R}_{q3}^{-1} \bar{R}_{2q})
\end{aligned}
$$

has a solution such that $\hat{A}_q - \hat{L}_q \hat{R}_q^{-1} \hat{C}$ is asymptotically stable and

$$ Z_q \geq 0, \tag{24} $$

$$ \hat{R}_{q1} - \hat{R}_{q2} \hat{R}_{q3}^{-1} \hat{R}_{q2}^T < 0. \tag{25} $$

In this case, a controller that achieves the objective is given by

$$ qx = \hat{A}_q x + B_{2q} u + \hat{L}_{q2} \hat{R}_{q3}^{-1} (y - \hat{C}_2 x), \tag{26} $$

$$ V_{q12} u = -\hat{C}_1 x - \hat{R}_{q2} \hat{R}_{q3}^{-1} (y - \hat{C}_2 x) \tag{27} $$

or equivalently

$$ qx = A_c x + B_c y, \tag{28} $$

$$ u = C_c x + D_c y \tag{29} $$

with

$$ A_c = \hat{A}_q - B_{2q} V_{q12}^{-1} \hat{C}_1 + B_{2q} V_{q12}^{-1} \hat{R}_{q2} \hat{R}_{q3}^{-1} \hat{C}_2 - \hat{L}_{q2} \hat{R}_{q3}^{-1} \hat{C}_2, \tag{30} $$

$$ B_c = -B_{2q} V_{q12}^{-1} \hat{R}_{q2} \hat{R}_{q3}^{-1} + \hat{L}_{q2} \hat{R}_{q3}^{-1}, \tag{31} $$

$$ C_c = -V_{q12}^{-1} \hat{C}_1 + V_{q12}^{-1} \hat{R}_{q2} \hat{R}_{q3}^{-1} \hat{C}_2, \tag{32} $$

$$ D_c = -V_{q12}^{-1} \hat{R}_{q2} \hat{R}_{q3}^{-1} \tag{33} $$

where $\hat{R}_{q1}$, $\hat{R}_{q2}$,$\hat{R}_{q3}$, and $\hat{L}_{q1}$, $\hat{L}_{q2}$ are given by the partitions,

$$ \begin{bmatrix} \hat{R}_{q1} & \hat{R}_{q2} \\ \hat{R}_{q2}^T & \hat{R}_{q3} \end{bmatrix} = \hat{R}_q, \qquad \begin{bmatrix} \hat{L}_{q1} & \hat{L}_{q2} \end{bmatrix} = \hat{L}_q. $$

Proof: See (Green and Limebeer 1995).

Remark 1. Considering the second Riccati equation (22), as $\Delta \to 0$, we obtain the following paradoxical results which hold irrespective of the underlying continuous-time systems.

$$ \lim_{\Delta \to 0} \hat{B}_q \hat{J} \hat{B}_q^T = 0, \qquad \lim_{\Delta \to 0} \hat{R}_q = \infty, \qquad \lim_{\Delta \to 0} \hat{A}_q = I. \tag{34} $$

This suggests that even with nonzero but small sampling period, numerical ill-conditioning is expected. In a similar fashion, it can be seen that the first Riccati equation (19) will experience numerical problems as $\Delta \to 0$. This type of numerical ill-conditioning has been previously observed in H_2 Riccati equations (Goodwin and Middleton 1992, Middleton and Goodwin 1990).

4. Discrete-time H_∞ Control Using the Delta-Domain Representation

For the discrete-time plant (10)-(12) in delta form, the H_∞ design problem seeks a causal, linear, time-invariant, finite-dimensional controller $K(\gamma)$ such that the closed-loop system is internally stable and for some $\gamma > 0$ satisfies

$$\|G_{z_p w}(P(\gamma), K(\gamma))\|_\infty < \gamma. \tag{35}$$

Here $P(\gamma)$ is the transfer function representation of (10)-(12) and $G_{z_p w}(P(\gamma), K(\gamma))$ is given by the following lower LFT,

$$G_{z_p w}(P(\gamma), K(\gamma)) = P_{11}(\gamma) + P_{12}(\gamma)K(\gamma)(I + P_{22}(\gamma)K(\gamma))^{-1}P_{21}(\gamma). \tag{36}$$

It should be noted that symbol γ is used to represent both the delta-domain variable and the H_∞ upper bound throughout this paper and its meaning is clear from context.

The following theorem provides necessary and sufficient conditions for the existence of such a delta form, discrete-time dynamic controller and characterizes one such controller.

Theorem 2. Consider the system given by equations (10)-(12) and suppose the following assumptions hold:

1. $(A_\delta, B_{2\delta}, C_2)$ is stabilizable and detectable,
2. $D_{12}^T D_{12} > 0$ and $D_{21} D_{21}^T > 0$,
3. $\operatorname{rank}\begin{bmatrix} A_\delta - \frac{e^{j\omega\Delta}-1}{\Delta} I & B_{2\delta} \\ C_1 & D_{12} \end{bmatrix} = n + m$, $\omega > 0$,
4. $\operatorname{rank}\begin{bmatrix} A_\delta - \frac{e^{j\omega\Delta}-1}{\Delta} I & B_{1\delta} \\ C_2 & D_{21} \end{bmatrix} = n + q$, $\omega > 0$.

Then a causal, linear, finite-dimensional controller exists which guarantees closed-loop internal stability and satisfies the H_∞ norm requirement (35) if and only if the following two conditions hold.

1. The Riccati equation

$$\begin{aligned} 0 \;=\; & \bar{Q}_\delta + A_\delta^T X_\delta + X_\delta A_\delta + \Delta A_\delta^T X_\delta A_\delta - [\bar{L}_\delta + B_\delta^T X_\delta(\Delta A_\delta + I)]^T \\ & [\bar{R}_\delta + \Delta B_\delta^T X_\delta B_\delta]^{-1}[\bar{L}_\delta + B_\delta^T X_\delta(\Delta A_\delta + I)] \end{aligned} \tag{37}$$

where

$$\begin{aligned}
\bar{R}_\delta + \Delta B_\delta^T X_\delta B_\delta &\triangleq \bar{R}_q/\Delta, \\
\bar{L}_\delta + B_\delta X_\delta(\Delta A_\delta + I) &\triangleq \bar{L}_q/\Delta, \\
\bar{Q}_\delta &\triangleq \bar{C}^T \bar{J} \bar{C}/\Delta
\end{aligned}$$

has a solution such that

$$A_\delta - B_\delta(\bar{R}_\delta + \Delta B_\delta^T X_\delta B_\delta)^{-1}(\bar{L}_\delta + B_\delta X_\delta(\Delta A_\delta + I))$$

is asymptotically stable and

$$X_\delta \geq 0, \tag{38}$$

$$R_{\delta 1} - R_{\delta 2}^T R_{\delta 3}^{-1} R_{\delta 2} < 0 \tag{39}$$

where $R_{\delta 1}, R_{\delta 2}, and R_{\delta 3}$ are given by compatible partition of $\bar{R}_\delta + \Delta B_\delta^T X_\delta B_\delta$.

2. The Riccati equation

$$\begin{aligned}
0 = {} & \hat{Q}_\delta + \hat{A}_\delta Z_\delta + Z_\delta \hat{A}_\delta^T + \Delta \hat{A}_\delta Z_\delta \hat{A}_\delta^T - [\hat{L}_\delta + (\Delta \hat{A}_\delta + I) Z_\delta \hat{C}^T] \\
& [\hat{R}_\delta + \Delta \hat{C} Z_\delta \hat{C}^T]^{-1} [\hat{L}_\delta + (\Delta \hat{A}_\delta + I) Z_\delta \hat{C}^T]^T
\end{aligned} \tag{40}$$

where

$$\begin{aligned}
\hat{R}_\delta + \Delta \hat{C} Z_\infty \hat{C}^T &\triangleq \Delta \hat{R}_q, \\
\hat{L}_\delta + (\Delta \hat{A}_\delta + I) Z_\infty \hat{C}^T &\triangleq \hat{L}_q, \\
\hat{Q}_\delta &\triangleq \hat{B} \hat{J} \hat{B}^T/\Delta
\end{aligned}$$

and

$$\left[\begin{array}{c|c} \hat{A}_\delta & \hat{B}_\delta \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{c|cc} A_\delta - B_{1\delta} R_{\delta d}^{-1} L_{\delta d} & B_1 \delta V_{\delta 21}^{-1} & 0 \\ \hline V_{\delta 12} R_{\delta 3}^{-1}(L_{\delta 2} - R_{\delta 2} R_{\delta d}^{-1} L_{\delta d}) & V_{\delta 12} R_{\delta 3}^{-1} R_{\delta 2} V_{\delta 21}^{-1} & I \\ C_2 - D_{21} R_{\delta d}^{-1} L_{\delta d} & D_{21} V_{\delta 21}^{-1} & 0 \end{array}\right], \tag{41}$$

where

$$\begin{aligned}
R_{\delta d} &= R_{\delta 1} - R_{\delta 2}^T R_{\delta 3}^{-1} R_{\delta 2}, \\
L_{\delta d} &= L_{\delta 1} - R_{\delta 2}^T R_{\delta 3}^{-1} L_{\delta 2}, \\
V_{\delta 12}^T V_{\delta 12} &= R_{\delta 3}, \\
V_{\delta 21}^T V_{\delta 21} &= -\gamma^{-2}(R_{\delta 1} - R_{\delta 2}^T R_{\delta 3}^{-1} R_{\delta 2}),
\end{aligned}$$

and $L_{\delta 1}, L_{\delta 2}$ are compatible partitions of $\bar{L}_\delta + B_\delta X_\delta(\Delta A_\delta + I)$, has a solution such that

$$\hat{A}_\delta - [\hat{L}_\delta + (\Delta \hat{A}_\delta + I)Z_\delta \hat{C}^T](\hat{R}_\delta + \Delta \hat{C} Z_\delta \hat{C}^T)^{-1}\hat{C}$$

is asymptotically stable and

$$Z_\infty \ \geq \ 0, \tag{42}$$

$$\tilde{R}_{\delta 1} - \tilde{R}_{\delta 2}\tilde{R}_{\delta 3}^{-1}\tilde{R}_{\delta 2}^T \ < \ 0. \tag{43}$$

In this case, a dynamic controller that achieves the objectives is given by

$$\delta x \ = \ \hat{A}_\delta x + B_{2\delta} u + \tilde{L}_{\delta 2}\tilde{R}_{\delta 3}^{-1}(y - \hat{C}_2 x), \tag{44}$$

$$V_{\delta 12} u \ = \ -\hat{C}_1 x - \tilde{R}_{\delta 2}\tilde{R}_{\delta 3}^{-1}(y - \hat{C}_2 x) \tag{45}$$

or equivalently

$$\delta x \ = \ A_c x + B_c y, \tag{46}$$

$$u \ = \ C_c x + D_c y \tag{47}$$

with

$$A_c \ = \ \hat{A}_\delta - B_{2\delta} V_{\delta 12}^{-1}\hat{C}_1 + B_{2\delta} V_{\delta 12}^{-1}\tilde{R}_{\delta 2}\tilde{R}_{\delta 3}^{-1}\hat{C}_2 - \tilde{L}_{\delta 2}\tilde{R}_{\delta 3}^{-1}\hat{C}_2, \tag{48}$$

$$B_c \ = \ -B_{2\delta} V_{\delta 12}^{-1}\tilde{R}_{\delta 2}\tilde{R}_{\delta 3}^{-1} + \tilde{L}_{\delta 2}\tilde{R}_{\delta 3}^{-1}, \tag{49}$$

$$C_c \ = \ -V_{\delta 12}^{-1}\hat{C}_1 + V_{\delta 12}^{-1}\tilde{R}_{\delta 2}\tilde{R}_{\delta 3}^{-1}\hat{C}_2, \tag{50}$$

$$D_c \ = \ -V_{\delta 12}^{-1}\tilde{R}_{\delta 2}\tilde{R}_{\delta 3}^{-1} \tag{51}$$

where $\tilde{R}_{\delta 1}$, $\tilde{R}_{\delta 2}$,$\tilde{R}_{\delta 3}$, and $\tilde{L}_{\delta 1}$, $\tilde{L}_{\delta 2}$ are given by the partitions,

$$\begin{bmatrix} \tilde{R}_{\delta 1} & \tilde{R}_{\delta 2} \\ \tilde{R}_{\delta 2}^T & \tilde{R}_{\delta 3} \end{bmatrix} = \hat{R}_\delta + \Delta \hat{C} Z_\delta \hat{C}^T, \qquad \begin{bmatrix} \tilde{L}_{\delta 1} & \tilde{L}_{\delta 2} \end{bmatrix} = \hat{L}_\delta + (\Delta \hat{A}_\delta + I) Z_\delta \hat{C}^T.$$

Proof. As in Theorem 1, assumption 1 is required for the existence of stabilizing delta controllers. Assumption 2 is identical to that in Theorem 1 since matrices D_{12} and D_{21} are the same in both forward-shift form and delta form. Assumptions 3 and 4 follows from assumptions 3 and 4 in Theorem 1 respectively by noting that the discrete-time, delta-domain frequency variable is $\gamma = \frac{e^{j\omega\Delta} - I}{\Delta}$ (Feuer and Goodwin 1996). The proof is then completed by substituting into Theorem 1 the relations and definitions

$$A_q = \Delta A_\delta + I, \qquad B_q \ = \ \Delta B_\delta,$$

$$\begin{aligned}
\hat{A}_q = \Delta\hat{A}_\delta + I, \qquad \hat{B}_q &= \Delta\hat{B}_\delta, \\
\bar{R}_\delta + \Delta B_\delta^T X_\delta B_\delta &\triangleq \bar{R}_q/\Delta, \\
\bar{L}_\delta + B_\delta X_\delta(\Delta A_\delta + I) &\triangleq \bar{L}_q/\Delta, \\
\bar{Q}_\delta &\triangleq \bar{C}^T \bar{J}\bar{C}/\Delta, \\
\hat{R}_\delta + \Delta\hat{C} Z_\infty \hat{C}^T &\triangleq \Delta\hat{R}_q, \\
\hat{L}_\delta + \hat{A} Z_\infty \hat{C}^T &\triangleq \hat{L}_q, \\
\hat{Q}_\delta &\triangleq \hat{B}\hat{J}\hat{B}^T/\Delta
\end{aligned}$$

and performing some rather tedious algebraic manipulations. □

The solution of Riccati equation (37) can be computed by using the associated delta-domain Hamiltonian matrix,

$$\bar{M} = \begin{bmatrix} A_m + \Delta B_\delta \bar{R}_\delta^{-1} B_\delta^T (I + \Delta A_m^T)^{-1} Q_m & -B_\delta \bar{R}_\delta^{-1} B_\delta (I + \Delta A_m^T)^{-1} \\ -(I + \Delta A_m^T)^{-1} Q_m & -(I + \Delta A_m^T)^{-1} A_m^T \end{bmatrix} \tag{52}$$

where

$$\begin{aligned}
A_m &= A_\delta - B_\delta \bar{R}_\delta^{-1} \bar{L}_\delta, \\
Q_m &= \bar{Q}_\delta - \bar{L}_\delta^T \bar{R}_\delta^{-1} \bar{L}_\delta
\end{aligned}$$

while the solution of Riccati equation (40) can be computed by using the associated Hamiltonian matrix,

$$\hat{M} = \begin{bmatrix} \hat{A}_m + \Delta\hat{C}^T \hat{R}_\delta^{-1} \hat{C} (I + \Delta\hat{A}_m^T)^{-1} \hat{Q}_m & -\hat{C}^T \hat{R}_\delta^{-1} \hat{C} (I + \Delta\hat{A}_m^T)^{-1} \\ -(I + \Delta\hat{A}_m^T)^{-1} \hat{Q}_m & -(I + \Delta\hat{A}_m^T)^{-1} \hat{A}_m^T \end{bmatrix} \tag{53}$$

where

$$\begin{aligned}
\hat{A}_m &= \hat{A}_\delta - \hat{L}_\delta \hat{R}_\delta^{-1} \hat{C} \\
\hat{Q}_m &= \hat{Q}_\delta - \hat{L}_\delta \hat{R}_\delta^{-1} \hat{L}_\delta^T.
\end{aligned}$$

Remark 2. It is possible to rephrase condition (2) in terms of the existence of a solution Y_δ to a filter Riccati equation satisfying requirements dual to (38) and (39). As in the continuous-time case, substituting Z_δ with Y_δ means that the spectral radius condition $\rho(X_\delta Y_\delta) < \gamma^2$ must be satisfied. Furthermore, $Z_\delta = Y_\delta(I - \gamma^{-2} X_\delta Y_\delta)^{-1}$ and the controller equations can be rewritten in terms of X_δ and Y_δ. In (Iglesias and Glover 1991), a set of discrete-time H_∞ controller design equations were developed by using this approach.

5. Unification of the Continuous-time and Discrete-Time H_∞ Theory

As discussed in the introduction, H_∞ design theory has been developed in both the continuous-time and discrete-time (the forward-shift form) domain. But due to the nature of the q-domain representations, explicit connections between continuous-time and discrete-time design theories have not been unified except by using the bilinear transformations. In this section, it is shown that the continuous-time H_∞ design equations are exactly recovered from the discrete-time δ-operator H_∞ design equations when the sampling period Δ is set to zero and which makes it possible to use the same set of design equations for both continuous-time and discrete-time design.

The exposition begins by citing the following theorem that gives necessary and sufficient conditions of the existence of an admissible continuous-time controller to the continuous-time system (1)-(3).

Theorem 3. (Green and Limebeer 1995, Stoovogel 1992b). Consider the continuous-time system given by equations (1)-(3) and assume the following assumptions hold:

1. (A, B_2, C_2) is stabilizable and detectable,
2. $D_{12}^T D_{12} = I$ and $D_{21} D_{21}^T = I$,
3. $\text{rank}\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m$, for all real ω,
4. $\text{rank}\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + q$, for all real ω,
5. $D_{11} = 0$ and $D_{22} = 0$.

Then a causal, linear, finite-dimensional controller exists which guarantees closed-loop internal stability and satisfies the H_∞ norm requirement

$$\|G_{z_p w}(P(s), K(s))\|_\infty < \gamma \tag{54}$$

where $P(s)$ is the transfer function representation of (1)-(3) and $G_{z_p w}(P(s), K(s))$ is given by the following lower LFT

$$G_{z_p w}(P(s), K(s)) = P_{11}(s) + P_{12}(s)K(s)(I + P_{22}(s)K(s))^{-1}P_{21}(s), \tag{55}$$

if and only if the following two conditions hold.

1. The Riccati equation

$$0 = X_\infty A_m + A_m^T X_\infty + Q_m - X_\infty (B_2 B_2^T - \gamma^{-2} B_1 B_1^T) X_\infty \tag{56}$$

where

$$\begin{aligned} A_m &= A - B_2 D_{12}^T C_1, && (57) \\ Q_m &= C_1^T (I - D_{12} D_{12}^T) C_1 && (58) \end{aligned}$$

has a solution such that $A_m - (B_2 B_2^T - \gamma^{-2} B_1 B_1^T) X_\infty$ is asymptotically stable and $X_\infty \geq 0$.

2. The Riccati equation

$$0 = \hat{A}_m Z_\infty + Z_\infty \hat{A}_m^T + \hat{Q}_m - Z_\infty (C_{2Z}^T C_{2Z} - \gamma^{-2} F_\infty^T F_\infty) Z_\infty \tag{59}$$

where

$$\begin{aligned} \hat{A}_m &= A + \gamma^{-2} B_1 (I - D_{21}^T D_{21}) B_1^T X_\infty - B_1 D_{21}^T C_2, && (60) \\ \hat{Q}_m &= B_1 (I - D_{21}^T D_{21}) B_1^T, && (61) \\ C_{2Z} &= C_2 + \gamma^{-2} D_{21} B_{1\delta}^T X_\infty, && (62) \\ F_\infty &= C_1^T D_{12} + X_\infty B_{2\delta} && (63) \end{aligned}$$

has a solution such that $\hat{A}_m - Z_\infty (C_{2Z}^T C_{2Z} - \gamma^{-2} F_\infty^T F_\infty)$ is asymptotically stable and $Z_\infty \geq 0$.

In this case, the central dynamic H_∞ controller is given by $K(s) \sim \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$ with

$$\begin{aligned} A_c &= A + B_1 B_1^T X_\infty - B_2 (D_{12}^T C_1 + B_2^T X_\infty) && (64) \\ &- [B_1 D_{21}^T + Z_\infty (C_2^T + \gamma^{-2} X_\infty B_1 D_{21}^T)](C_2 + \gamma^{-2} D_{21} B_1^T X_\infty) \\ B_c &= B_1 D_{21}^T + Z_\infty (C_2^T + \gamma^{-2} X_\infty B_1 D_{21}^T) && (65) \\ C_c &= -(D_{12}^T C_1 + B_2^T X_\infty) && (66) \end{aligned}$$

It is desired to prove that when the sampling period Δ is set to zero, the delta-domain theorem (Theorem 2) is equivalent to the continuous-time theorem (Theorem 3) under identical system assumptions. Note that there are some differences between the assumptions of these two theorems. In order to show the equivalence, we make further assumptions

$$D_{12}^T D_{12} = I, \quad D_{21} D_{21}^T = I, \quad D_{11} = 0, \quad D_{22} = 0 \tag{67}$$

to Theorem 2 which result in identical system assumptions to both Theorem 2 and Theorem 3.

Theorem 4. If the sampling period Δ is set to zero and (67) holds in Theorem 2, then Theorem 2 and Theorem 3 are equivalent.

Proof. In order to show the equivalence of Theorem 2 and Theorem 3, we need to show that the assumptions made to both theorems are the same which is obvious from previous discussions and (67). It remains to be shown that the two conditions in both theorems and the controllers given in both theorems are exactly the same when the sampling period is set to zero.

Note that delta-domain Hamiltonian matrices (52) and (53) corresponding respectively to the Riccati equations (37) and (40), can be rewritten as

$$\bar{M} = \begin{bmatrix} I & \Delta B_\delta \bar{R}_\delta^{-1} B_\delta^T \\ 0 & I + \Delta A_m \end{bmatrix}^{-1} \begin{bmatrix} A_m & -B_\delta \bar{R}_\delta^{-1} B_\delta^T \\ -Q_m & -A_m^T \end{bmatrix} \tag{68}$$

and

$$\hat{M} = \begin{bmatrix} I & \Delta \hat{C}^T \hat{R}_\delta^{-1} \hat{C} \\ 0 & I + \Delta \hat{A}_m \end{bmatrix}^{-1} \begin{bmatrix} \hat{A}_m^T & -\hat{C}^T \hat{R}_\delta^{-1} \hat{C} \\ -\hat{Q}_m & -\hat{A}_m \end{bmatrix} \tag{69}$$

which for $\Delta = 0$ collapse to

$$\bar{M} = \bar{M}_c \triangleq \begin{bmatrix} A_m & -B_\delta \bar{R}_\delta^{-1} B_\delta \\ -Q_m & -A_m^T \end{bmatrix}, \qquad \hat{M} = \hat{M}_c \triangleq \begin{bmatrix} A_m^T & -\hat{C}^T \hat{R}_\delta^{-1} \hat{C} \\ -\hat{Q}_m & -\hat{A}_m \end{bmatrix}. \tag{70}$$

Now, to show that condition 1 of Theorem 2 is equivalent to condition 1 of Theorem 3, observe that the three elements of Hamiltonian matrix $\bar{M}_c$ become

$$A_m = A_\delta - B_\delta \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ D_{12}^T C_1 \end{bmatrix} = A_\delta - B_{2\delta} D_{12}^T C_1, \tag{71}$$

$$Q_m = \bar{Q}_\delta - \begin{bmatrix} 0 & C_1^T D_{12} \end{bmatrix} \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ D_{12}^T C_1 \end{bmatrix} = C_1^T (I - D_{12} D_{12}^T) C_1, \tag{72}$$

$$B_\delta \bar{R}_\delta^{-1} B_\delta = \begin{bmatrix} B_{1\delta} & B_{2\delta} \end{bmatrix} \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} B_{1\delta}^T \\ B_{2\delta}^T \end{bmatrix} = B_{2\delta} B_{2\delta}^T - \gamma^{-2} B_{1\delta} B_{1\delta}^T. \tag{73}$$

It follows that since for $\Delta = 0$, $A_\delta = A$ and $B_\delta = B$, the Riccati equation (37) recovers the continuous-time Riccati equation (56). The asymptotical stability equivalence of $A_\delta - B_\delta(\bar{R}_\delta + \Delta B_\delta^T X_\delta B_\delta)^{-1}(\bar{L}_\delta + B_\delta X_\delta(\Delta A_\delta + I))$ with $A_m - (B_2 B_2^T - \gamma^{-2} B_1 B_1^T) X_\infty$ can be established by noting that

$$\bar{L}_\delta = \begin{bmatrix} 0 \\ D_{12}^T \end{bmatrix}, \qquad \bar{R}_\delta = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix} \tag{74}$$

and $A_\delta = A, B_\delta = B$ and $X_\delta = X_\infty$. Condition (39) is automatically satisfied since $R_{\delta 1} - R_{\delta 2}^T R_{\delta 3}^{-1} R_{\delta 2} = -\gamma^2 I$.

To show the equivalence of condition 2 in both theorems, the quantities involved in (41) need to be evaluated at $\Delta = 0$ which gives

$$\bar{R}_\delta = \begin{bmatrix} -\gamma^2 D_{12} D_{12}^T & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} R_{\delta 1} & R_{\delta 2} \\ R_{\delta 2}^T & R_{\delta 3} \end{bmatrix}, \tag{75}$$

$$\begin{bmatrix} L_{\delta 1} \\ L_{\delta 2} \end{bmatrix} = \begin{bmatrix} 0 & I \\ D_{12}^T & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 \end{bmatrix} \begin{bmatrix} C_1 \\ 0 \end{bmatrix} = \begin{bmatrix} B_{1\delta}^T X_\delta \\ D_{12}^T C_1 + B_{2\delta}^T X_\delta \end{bmatrix}, \tag{76}$$

$$R_{\delta d} = R_{\delta 1} - R_{\delta 2}^T R_{\delta 3}^{-1} R_{\delta 2} = -\gamma^2 I, \tag{77}$$

$$L_{\delta d} = L_{\delta 1} - R_{\delta 2}^T R_{\delta 3}^{-1} L_{\delta 2} = B_{1\delta}^T X_\delta, \tag{78}$$

$$V_{\delta 12} = V_{\delta 21} = I. \tag{79}$$

Then, realization (41) becomes

$$\left[\begin{array}{c|c} \hat{A}_\delta & \hat{B}_\delta \\ \hline \hat{C} & \hat{D} \end{array}\right] \triangleq \left[\begin{array}{c|cc} \hat{A}_\delta & \hat{B}_1\delta & \hat{B}_2\delta \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right] = \left[\begin{array}{c|cc} A_\delta + B_{1\delta} B_{1\delta}^T X_\delta & B_1\delta & 0 \\ \hline D_{12}^T C_1 + B_{2\delta}^T X_\delta & 0 & I \\ C_2 + \gamma^{-2} D_{21} B_{1\delta}^T X_\delta & D_{21} & 0 \end{array}\right]. \tag{80}$$

Hence the three elements of the Hamiltonian matrix $\hat{M}_c$ in (70) can be written as

$$\hat{A}_m = \hat{A}_\delta - \hat{L}_\delta \hat{R}_\delta^{-1} \hat{C} = A_\delta + \gamma^{-2} B_{1\delta}(I - D_{21}^T D_{21}) B_{1\delta}^T X_\delta - B_{1\delta} D_{21}^T C_2, \tag{81}$$

$$\hat{Q}_m = \hat{Q}_\delta - \hat{L}_\delta \hat{R}_\delta^{-1} \hat{L}_\delta^T = B_{1\delta}(I - D_{21}^T D_{21}) B_{1\delta}^T, \tag{82}$$

$$\begin{aligned} \hat{C}^T \hat{R}_\delta^{-1} \hat{C} &= (C_1^T D_{12} + X_\delta B_{2\delta})^T (C_1^T D_{12} + X_\delta B_{2\delta}) \\ &+ (C_2 + \gamma^{-2} D_{21} B_{1\delta}^T X_\delta)^T (C_2 + \gamma^{-2} D_{21} B_{1\delta}^T X_\delta). \end{aligned} \tag{83}$$

It follows that since for $\Delta = 0$, $A_\delta = A, B_\delta = B$, the Riccati equation (40) recovers the continuous-time Riccati equation (59). Also, note that for $\Delta = 0$

$$\hat{L}_\delta = \begin{bmatrix} 0 & B_{1\delta} D_{21}^T \end{bmatrix}, \qquad \hat{R}_\delta = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix}. \tag{84}$$

The asymptotical stability equivalence of $\hat{A}_\delta - [\hat{L}_\delta + (\Delta \hat{A}_\delta + I) Z_\delta \hat{C}^T](\hat{R}_\delta + \Delta \hat{C} Z_\delta \hat{C}^T)^{-1} \hat{C}$ with $\hat{A}_m - Z_\infty (C_{2Z}^T C_{2Z} - \gamma^{-2} F_\infty^T F_\infty)$ can be established by using substitution (84). In addition, condition (43) is obviously satisfied.

Finally, the continuous-time controller described by equations (64)-(66) can be obtained by substituting

$$\begin{bmatrix} \tilde{R}_{\delta 1} & \tilde{R}_{\delta 2} \\ \tilde{R}_{\delta 2}^T & \tilde{R}_{\delta 3} \end{bmatrix} = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix}, \tag{85}$$

$$\begin{bmatrix} \tilde{L}_{\delta 1} & \tilde{L}_{\delta 2} \end{bmatrix} = \begin{bmatrix} Z_\infty (C_1^T D_{12} + X_\delta B_{2\delta}^T) & B_{1\delta} D_{21}^T + Z_\infty (C_2^T + \gamma^{-2} X_\delta B_{1\delta} D_{21}^T) \end{bmatrix}, \tag{86}$$

and $V_{\delta 12} = I$ into equations (49)-(51) by noting the fact that at $\Delta = 0$, $A_\delta = A$, $B_{1\delta} = B_1$, $B_{2\delta} = B_2$, $X_\delta = X_\infty$, and $Z_\delta = Z_\infty$. □

6. Illustrative Example

To illustrate H_∞ controller design by using both the q-operator and δ-operator representations and to observe the behavior of the controllers as $\Delta \to 0$, we consider the two mass/spring benchmark system (Collins *et al.* 1997) given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1.25 & 1.25 & 0 & 0 \\ 1.25 & -1.25 & 0 & 0 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \ D_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ D_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ D_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \ D_{22} = 0.$$

A Matlab program, which can be used for both continuous-time and discrete-time controller design, has been developed based on the solution of the delta H_∞ controller problem given in Theorem 2. It should be noted that the word length used in Matlab is 32 bits which corresponds to 7 significant decimal digits.

Sample period	0.5 s	0.05 s	0.0005 s	0.00005 s
Forward-shift controller obtained by the bilinear transformation approach	0.0303±0.7405i -0.1003±0.2322i	0.9301±0.1727i 0.8001±0.0873i	0.9994±0.0018i 0.9978±0.0011i	0.9999±0.0002i 0.9998±0.0001i
Forward-shift controller obtained by the direct forward shift approach	-0.2821±0.6834i -0.0836 0.0260	0.9297±0.1732i 0.8000±0.0866i	0.9994±0.0018i 0.9978±0.0011i	ALGORITHM FAILED
Forward-shift controller obtained by the direct delta approach	-0.2821±0.6834i -0.0836 0.0260	0.9297±0.1732i 0.8000±0.0866i	0.9994±0.0018i 0.9978±0.0011i	0.9999±0.0002i 0.9998±0.0001i
Delta controller obtained by the direct delta approach	-2.5642±1.3668i -2.1672 -1.9481	-1.4051±3.4632i -4.0010±1.7328i	-1.1229±3.6779i -4.3333±2.1466i	-1.1201±3.6797i -4.3365±2.1508i

Table 1: Pole locations of the controllers

As seen in Table 1, which shows the controller poles, a set of discrete-time stabilizing controllers for the 3rd-order plant have been obtained by using the bilinear transform approach (row 2), the direct forward-shift approach (row 3) and the direct delta approach derived in this paper (rows 4 and 5). It should be noted that the forward-shift controller shown in row 4 is obtained by first computing the delta-domain controller and then transforming back to the forward-shift form. It can be seen from rows 3 and 4 of Table 1 that, as expected, for the latter 2 approaches, the controllers obtained are identical for any sampling period Δ. However, due to numerical ill-conditioning, discrete-time controllers based on the forward-shift approach cannot be obtained when the sampling period becomes sufficiently small. In this example, the critical value is $\Delta = 0.00005$ seconds. Furthermore, as the sampling period decreases, each pole of the discrete-time controller represented in forward-shift form approaches the unit circle regardless of whether the controller is obtained by the direct forward-shift algorithm, the bilinear transformation approach, or by transforming the directly designed delta controller to forward-shift form. This clearly leads to sensitivity to roundoff errors when the controllers are implemented.

However, by using the delta approach, it can be seen that as $\Delta \to 0$, the discrete-time controller converges to the continuous-time controller as stated in previous sections. The continuous-time controller and the delta controller for $\Delta = 10^{-6}$ seconds are given by

$$K(s) = \frac{296.3968(s+0.5752)[(s-0.1954)^2 + 1.4639^2]}{[(s+4.3369)^2 + 2.1512^2][(s+1.1198)^2 + 3.6799^2]}$$

and

$$K(\gamma) = \frac{296.3891(\gamma+0.5752)[(\gamma-0.1954)^2 + 1.4639^2]}{[(\gamma+4.3368)^2 + 2.1512^2][(\gamma+1.1198)^2 + 3.6799^2]}$$

respectively.

It should be noted that there are some differences between the controller poles obtained using the bilinear transformation approach (row 1) and the direct forward-shift or delta operator approach (rows 2 through 4). This is because the *central* continuous-time H_∞ controller of the continuous-time system obtained by a bilinear transformation of the original discrete-time system does not necessarily correspond to the *central* discrete-time H_∞ controller. For $\Delta = 0.5s$, the forward-shift controllers obtained by direct forward-shift design and bilinear transformation approaches are given respectively by

$$K(z) = \frac{15.4418(z-0.7557)[(z-0.8071)^2 + 0.7454^2]}{(z+0.0836)(z-0.0260)[(z+0.2821)^2 + 0.6834^2]}$$

and

$$K(z) = \frac{7.6467(z+1.0000)(z-0.7485)[(z-0.8316)^2 + 0.7047^2]}{[(z-0.0303)^2 + 0.7405^2][(z+0.1003)^2 + 0.2322^2]}.$$

7. Conclusion

This paper has developed a method to directly design delta operator, discrete-time H_∞ controllers. The design equations converge to the corresponding continuous-time design equations as the sampling period decreases. This provides a unification of the continuous-time and discrete-time theories. This approach avoids the numerical ill-conditioning experienced by algorithms based on the forward-shift representation of discrete-time systems. In addition, the delta controllers are less sensitive to implementation roundoff errors than forward-shift controllers at small sample periods. The results were demonstrated by a numerical example.

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