MICROROBOTS AND MICROMECHANICAL SYSTEMS

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Abstract

The domain of micromechanical systems is an extensive, useful and yet largely unexplored area. It is likely that, in many applications, small mechanisms will prove to be faster, more accurate, gentler and less expensive than the macro systems presently used. This paper explores the advantages of micromechanical systems and analyzes the scaling of forces in the micro domain.

1. Introduction

Over the past 20 years we have witnessed a revolution in the miniaturization of electronic components from the dimensions of inches to microns. Now we have the potential to accomplish a similar revolution with many mechanical components [1]. Moreover, as we shrink various devices, fundamental scaling properties enhance the performance of these small systems. In this paper we shall examine what happens as we scale the dimensions of motors and other mechanical systems into the micro domain. (An excellent overview of the micro field is given in an NSF report on microdynamics [2].)

There are several reasons for developing micro systems. Systems with small dimensions have advantages in handling small parts, advantages that include speed, accuracy and gentleness. Intellectually, one is also drawn to this extensive and relatively unexplored domain. Benefits will probably accrue in such areas as electronics assembly, medicine and space exploration. Micro systems potentially have numerous advantages in performing micro tasks that are at, or even beyond, the limits of human manipulation and patience.

The actuators we use in the macro world often depend upon magnetic forces. As we shall see, these magnetic forces scale poorly into the micro domain. Fortunately, several other forces scale well. Scaling theory shows that electrostatics, pneumatics, surface tension and biological forces are all strong enough to be useful in the small domain.

This paper explores the field of microrobotic and micromechanical systems. The first portion of the paper examines the advantages, extent and applicability of micro systems. The second portion analyzes what type of
forces will be useful to power these micromechanical systems. ‘Microrobotic’
is used to refer to small, automated systems; ‘micro teleoperator’ refers to
a small system that is remotely controlled by a human; and ‘micromechani-
cal systems’ refers to all small mechanical systems.

2. Advantages of micromechanical systems

This Section discusses what sizes of mechanical systems are appropriate
to handle small objects. For example, what sized tooling is appropriate for
assembling optical fibers, lasers and light detectors used in fiber optic com-
munication. There are advantages in using micro tooling that is more com-
mensurate in size with these small parts than the macro tooling presently
used. These advantages include:

(a) Higher throughput. The time for an operation scales with the
system’s linear dimension raised to roughly the first power, $s^1$. That is, a
system one-tenth the size of the original has the potential of performing
the task ten times faster. (Later in the analysis, it will be shown that the
exponent ranges from 0.5 to 1.5 for most force laws.)

(b) Higher accuracy. Problems due to phenomena such as the tempera-
ture coefficient of expansion, deflection and vibration become less trouble-
some as the size of the system decreases. Despite these advantages, it is
curious that people generally consider smaller mechanical systems to be less
accurate, indeed often toys. This need not be the case; the precision with
which small machines can be made may ultimately be limited by atomic
dimensions [3].

(c) Gentleness. The small forces and masses associated with small
systems make them more gentle.

(d) Improved performance. Because the cost of materials scales as $s^3$
(the volume), a small system can be built with very expensive materials that
have desirable properties. This freedom to use exotic materials often allows
the designer to improve performance.

(e) Floor space. An important consideration in manufacturing plants is
the cost of floor space. Small systems reduce this cost and may also enable
an operator to tend more machines because of their proximity.

Well-made mechanical systems tend to have an accuracy (deviation
from the ideal) and resolution (least motion increment) of a part in $10^5$, give
or take an order of magnitude or two. To obtain the precision movements
usually needed to handle small parts, macro designers resort to devices such
as massive granite tables and temperature controls. Not only do these mas-

sive mechanical systems need to be made with extreme precision, but they
also tend to be slow. In contrast, a centimeter-sized system with a similar
part in $10^5$ resolution has an incremental motion of a tenth of a micron,
and it is potentially faster handling millimeter-sized components.

Electronics exemplifies a field where numerous advantages have accrued
from miniaturization. The small size of VLSI electronic circuits, for example,
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(deviation in \( 10^5 \), gives movements devices such as these machines, but they are smaller than a micron, have accrued or example, allows designers to use exotic and expensive materials having desirable properties. A case in point: while one can use gold for interconnecting VLSI components, it would be prohibitively expensive for discrete components. Another advantage is convenience: if the word processor I am using were made of vacuum tubes, it would fill my office instead of a corner of my desk. There are also speed advantages in moving the electronic components closer together. Thus, I believe that the advantages in micro structures, which now seem so obvious in electronics, will someday be obvious in micromechanical systems as well.

Personal experiences have led me to an appreciation of using small mechanical systems to handle small parts. A system on which I collaborated used a meter-sized robot to handle millimeter-sized chips with an accuracy of about ten microns. This macro system had several disadvantages. The forces needed to move the robot's mass were much larger than the forces that would destroy the chip, and the robot could easily crush the chip. A lot of time was spent building fixtures that was compliant yet still maintained the high accuracy needed. A one centimeter robot would have about one millionth the mass, and hence require forces roughly a millionth of those needed to move the macro robot. Temperature changes of the macro robot were also troublesome. A one degree change of temperature led to about 10 microns expansion, the total error budget of our task. The larger robot also spent most of the time moving the chip from one work station to another. Because of the size of the robot and associated equipment, these workstations had to be several feet apart. If a centimeter-sized robot and associated equipment could have been used, the distances and transit times could have been dramatically reduced. Finally, finding space for this robot and its four by six foot table in a clean room was difficult. This thousand-to-one difference between the size of the macro robot and the chips is equivalent to using a bulldozer to move sugar cubes. The corresponding accuracy requirement is equivalent to positioning the sugar cube to within a hair's width. On the larger scale of the bulldozer, it is easier to see the advantages of using tooling commensurate with the parts being handled.

3. The large and small

The domain of small mechanical systems is an extensive area ripe for exploration. There are many intriguing things to learn and useful properties to discover [4]. To help display the range of sizes of mechanical systems, those available to mankind are logarithmically plotted in Fig. 1. A logarithmic plot has been chosen because each time the scale of the system under investigation is changed by several orders of magnitude, there are new phenomena to study and exploit. The scale shows the size of objects in Ångstroms \( (10^{-10} \text{ m}) \). Atoms, which are probably the smallest mechanical particles we will use in the near future, are roughly 1 Å in diameter. Man is about \( 10^{10} \text{ Å} \) (meter sized), and the universe about \( 10^{37} \text{ Å} \) in diameter. For
convenience, this scale is divided into the micro domain, which contains those systems 10 cm in size and smaller, and the macro domain, which contains meter-sized and larger systems. The micro domain represents about a quarter of the total scale. However, since mechanical systems larger than a few miles long probably will not be built in the near future, the micro domain represents the major portion of the scale presently available for investigation. The vertical dotted lines delineate the normal range of mechanical systems. The whimsical photograph (Fig. 2) of an ant holding a 900 micron silicon gear helps to challenge our preconceptions graphically. The gear was made by Gabriel, Mehregany and Trimmer [5].
4. Small mechanical systems of interest

In building small robots and systems, one is faced with problems similar to those faced by early macro machine makers. One cannot easily obtain a micro motor, gear or screw. Building these systems will take ingenuity. Given the large number of things that can be done, however, an important early task is deciding what systems have the most utility, and the greatest chance of success. Several areas seem especially promising: electronics assembly, surgery, space exploration and micromachining.

Electronics assembly is becoming increasingly difficult for macro tooling for at least three reasons. First, new generations of electronics generally require more interconnections. Secondly, to reduce the propagation delay between devices, designers are moving components closer together, and they are decreasing the inter-device capacitance by reducing the size of the interconnecting lands and pads. Thirdly, many of the new
technologies and materials make the devices more susceptible to damage. Smaller tooling excels in handling these difficulties, because it is inherently faster, more accurate and gentler. Many electronics assembly tasks could be performed nicely on a tea saucer if millirobotic systems were available. The desire to make more complex miniature circuits will surely challenge assembly concepts.

Despite the many advances in medicine, many operations still require opening a large wound in the patient’s body. Yet the actual repair task usually does not require an opening this large. The development of minute actuators, sensors, and other mechanical parts will help facilitate building micro manipulators and teleoperators that can be inserted into the body through a small incision or needle.

Space exploration is limited by the ability to move mass out of the earth’s gravitational field. Reducing the size, and hence the mass, of the mechanical systems will certainly increase the number of missions that can be performed.

New techniques are available to make micro structures, actuators and sensors [6 - 22]. Rapid progress in this area has been made on micro machining silicon using anisotropic etches developed for the electronics industry. Modifying these techniques makes it possible to fabricate very fine mechanical structures, and this silicon technology can be used to build the components needed for these micromechanical structures. For example, it is possible to build completely assembled mechanical structures on a silicon wafer [23, 24]. During the processing of the silicon wafer, structural and sacrificial layers are deposited on top of each other in selected areas. After processing, the sacrificial layers are etched away, leaving structural material in any desired shape. Using these techniques, complete gear boxes including the housing and supportingshafts can be built on silicon wafers. These nano gear boxes are typically half a millimeter in size. Several groups are also working on silicon-based microactuators to power these tiny devices [25 - 34]. While the technology is not presently available, one can envision making components on the atomic scale.

Should an industrial base capable of building micromachines be developed? Material costs (and possibly floor space costs) may be lower. One also assumes that the developers of new micro systems will take advantage of new automation and robotic techniques, and so micro assembly systems will be more efficient than the older macro systems they replace. What are the uses of micro capabilities? Some of the readers will probably help to answer this question.

5. The scaling of mechanical systems

When designing micromechanical systems, the first thing to decide is what forces should be used for activation. How the different forces scale into the micro in the App

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the micro domain is described below. The details of this analysis are given in the Appendix.

A matrix formalism is used to describe the scaling results. This nomenclature is designed to show a number of different cases and scale sizes in a simple format. The size of the system is represented by a single scale variable \( s \), which represents the linear scale of the system. The choice of \( s \) for a system is a bit arbitrary. \( s \) could be the separation between the plates of a capacitor, or it might be the length of one edge of the capacitor. Once chosen, however, it is assumed that all dimensions of the system are scaled down in size as \( s \) is decreased. Nominally \( s = 1 \); if \( s \) is then changed to 0.1, all the dimensions of the system are decreased by a factor of ten. A number of different cases are shown in one equation. For example,

\[
F = \begin{bmatrix} s^1 \\ s^2 \\ s^3 \\ s^4 \end{bmatrix}
\]

shows four cases for the force law. The top scales as \( s^1 \), next scales as \( s^2 \), etc. The scaling of the time required to perform an operation, as discussed later, is

\[
t = \begin{bmatrix} s^{1.5} \\ s^1 \\ s^{0.5} \\ s^0 \end{bmatrix}
\]

The top time scaling, \( s^{1.5} \), always refers to the case where force scales as \( s^1 \), etc. This notation is used consistently throughout this paper. A dash [\( - \)] means that this case does not apply. (Equation (2) is a general relationship that holds whenever the forces scale as in eqn. (1). This is shown below.)

### 5.1. Magnetic forces

This Section describes the scaling of magnetic forces into the micro domain. Three cases are examined: constant current density, constant heat flow per unit surface area and constant temperature rise across the windings. It is assumed that the magnetic forces are generated by the interaction of two wires carrying current. (The scaling results of the interaction between a wire and permanent magnet are given in braces \( \{s^n\} \).)

In the first case, the current density is assumed to be constant \([s^0]\), and hence a wire with one-tenth the cross-sectional area carries one-tenth the current. The heat generated per volume of windings is constant for this case. The force generated for this constant current case scales as \([s^4]\) \(\{s^3\}\), i.e., when the system decreases by a factor of ten in size, the force generated magnetically decreases by a factor of ten thousand. Clearly this is not a strong micro force.
Since heat can be more easily conducted out of a small volume, it is possible to run isolated small motors with higher current densities. However, increasing the current density makes the motors much less efficient. If the heat flow per unit surface area of the windings is constant, the current density in the wires scales as \([s^{-0.5}]\). This increase in current density for small systems increases the force generated, and the force scales as \([s^3] ((s^2))\).

A third possible constraint on the magnetic system as it is scaled down in size is the maximum temperature that the wire and insulation can withstand. If the system parameters are scaled so that there is a constant temperature difference between the windings and surrounding environment, then the current scales as \([s^{-1}]\), and the force scales as \([s^2] ((s^2))\). As will be discussed later, forces that scale as \([s^2]\) are useful in small systems. However, below there are described several other forces that also scale as \([s^2]\), which do not waste the large amounts of power dissipated by this magnetic case.

Using the matrix notation above, the currents required for the different force scaling laws are given by:

\[
J = \begin{bmatrix}
- \\
\frac{1}{s} \\
\frac{1}{s^{0.5}} \\
s^0
\end{bmatrix}
\]

In the case of thin films, current densities may be limited even more than assumed by the above analysis by electromigration phenomena.

5.2. Electrostatic forces

Electrostatic actuators have a distinguished history [35], but are not in general use for motors. Electrostatic forces, however, become significant in the micro domain and have numerous potential applications. The exact form of the scaling of electrostatic forces depends upon how the \(E\) field changes with size. Generally, the breakdown \(E\) field of insulators increases as the system becomes smaller [36, 37]. Two cases will be examined here: (1) constant \(E\) field \((E = [s^0])\); (2) an \(E\) field that increases slightly as the system becomes smaller \((E = [s^{-0.5}]\)).

For the constant electric field \((E = [s^0])\), the force scales as \([s^2]\). When \(E\) scales as \([s^{-0.5}]\), then the force has the even better scaling of \(F = [s^1]\).

When the size of the system is decreased, both of these force laws give higher accelerations and smaller transit times.

There are several other interesting forces. Biological forces from muscle are proportional to the cross-section of the muscle, and scale as \([s^2]\). Pneumatic and hydraulic forces are caused by pressures \((P)\) and also scale as \([s^2]\). Surface tension has an absolutely delightful scaling of \([s^1]\) because it depends upon the length of the interface.
5.3. The unit cube

Below is a discussion of how the above force laws affect the acceleration, transit time, power generation and power dissipation as one scales to smaller domains. In going from here to there as quickly as possible with a certain force, one wants to accelerate for half the distance, and then decelerate. The mass of the object scales as \([s^3]\) (density is assumed to be intensive, or to not change with scale). Now the equations of dynamics give

\[
\begin{align*}
a &= F/m = [s^F][s^{-3}] \\
t &= (2x/a)^{1/2} = (2xm/F)^{1/2} \\
t &= ([s^1][s^3][s^{-F}])^{1/2}
\end{align*}
\]

where \(s^F\) represents the scaling of the force \(F\). Here only the time to accelerate has been calculated, but an equal time is needed to decelerate, and both these times scale in the same way. For the forces given in eqn. (1), the accelerations and transit times can be expressed as

\[
a = \begin{bmatrix} s^{-2} \\ s^{-1} \\ s^0 \\ s^1 \\ s^{1.5} \\ s^1 \\ s^{0.5} \\ s^0 \end{bmatrix}
\]

\[
t = \begin{bmatrix} s^{-2} \\ s^{-1} \\ s^0 \\ s^1 \\ s^{1.5} \\ s^1 \\ s^{0.5} \\ s^0 \end{bmatrix}
\]

Even in the worst case, where \(F = [s^4]\), the time required to perform a task remains constant when the system is scaled down. Under more favorable force scaling, for example, the \(F = [s^2]\) scaling case, the time required decreases as \([s]\) with the scale. A system ten times smaller can perform an operation ten times faster. This is an observation that we know intuitively: small things tend to be quick.

Inertial forces tend to become insignificant in the small domain, and in many cases kinematics may replace dynamics. This leads to interesting control strategies.

5.4. Power generated and dissipated

As the scale of a system is changed, one wants to know how the power produced depends upon the force laws. For example, consider the unit cube above, which is first accelerated and then decelerated. The power, \(P\), or work done on the object per unit time is

\[
P = Fx/t
\]

The scaling of each of the terms on the right is known.
\[
P = \begin{bmatrix}
  s^1 \\
  s^2 \\
  s^3 \\
  s^4
\end{bmatrix}
\begin{bmatrix}
  s^1 \\
  s^1 \\
  s^1 \\
  s^1
\end{bmatrix}
\begin{bmatrix}
  s^{-1.5} \\
  s^{-1} \\
  s^{-0.5} \\
  s^0
\end{bmatrix}
\]

(9)

\[
P = \begin{bmatrix}
  s^{0.5} \\
  s^2 \\
  s^{3.5} \\
  s^5
\end{bmatrix}
\]

The power that can be produced per unit volume \((V = [s^3])\) is

\[
P \quad \frac{\rho}{V} = \begin{bmatrix}
  s^{-2.5} \\
  s^{-1} \\
  s^{0.5} \\
  s^2
\end{bmatrix}
\]

(10)

When the force scales as \([s^2]\), then the power per unit volume scales as \([s^{-1}]\). For example, when the scale decreases by a factor of ten, the power that can be generated per unit volume increases by a factor of ten. For force laws with a higher power than \(s^2\), the power generated per volume degrades as the scale decreases. There are several attractive force laws that behave as \(s^2\), and one should try to use these forces when designing small systems.

For the magnetic case, one may be concerned about the power dissipated by the resistive loss of the wires. The power due to this resistive loss, \(P_R\), is

\[
P_R = I^2R = (JA)^2 \frac{\rho l}{A}
\]

(11)

where \(A\) is the cross section of the wire, \(\rho\) is the resistivity of the wire, and \(l\) is the length of the wire. This gives

\[
P_R = J^2[Al] \rho
\]

(12)

where \(Al\) is the volume. The resistivity scales as \([s^0]\) and the volume scales as \([s^3]\), and from eqn. (3) above,

\[
J = \begin{bmatrix}
  - \\
  s^{-1} \\
  s^{-0.5} \\
  s^0
\end{bmatrix}
\]

(13)
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\[
P_R = \begin{bmatrix}
  - \\
  s^1 \\
  s^2 \\
  s^3 \\
\end{bmatrix}
\]  

(14)

\[
\frac{P_R}{V} = \begin{bmatrix}
  - \\
  s^{-2} \\
  s^{-1} \\
  s^0 \\
\end{bmatrix}
\]  

(15)

For the magnetic case where force scales as \([s^2]\), the power that must be dissipated per unit volume scales as \([s^{-2}]\), or, when the scale is decreased by a factor of ten, a hundred times as much power must be dissipated within the volume. This magnetic case is bad if one is concerned about power density or the amount of cooling needed. However, future advances in the field of superconductors may reduce or eliminate this problem.

5.5. Summary of the scaling results

The force has been found to scale in one of four different ways: \([s^1]\), \([s^2]\), \([s^3]\), \([s^4]\). If the scale size is decreased by a factor of ten, the forces for these different laws decrease by ten, one hundred, one thousand, and ten thousand respectively. Obviously, one wants to work with force laws that behave as \([s^1]\) or \([s^2]\). The different cases that lead to these force laws, the accelerations, the transit times and the power generated per unit volume are given below.

(a) \([s^1]\): surface tension; electrostatics where \(F = [s^{-0.5}]\).

(b) \([s^2]\): electrostatics where \(E = [s^0]\); pressure forces; biological forces; magnetics where \(J = [s^{-1}]\).

(c) \([s^3]\): magnetics where \(J = [s^{-0.5}]\).

(d) \([s^4]\): magnetics where \(J = [s^0]\).

\[
F = \begin{bmatrix}
  s^1 \\
  s^2 \\
  s^3 \\
  s^4 \\
\end{bmatrix}, \quad a = \begin{bmatrix}
  s^{-2} \\
  s^{-1} \\
  s^0 \\
  s^1 \\
\end{bmatrix}, \quad t = \begin{bmatrix}
  s^1.5 \\
  s^1 \\
  s^{0.5} \\
  s^0 \\
\end{bmatrix}, \quad \frac{P}{V} = \begin{bmatrix}
  s^{-2.5} \\
  s^{-1} \\
  s^{0.5} \\
  s^2 \\
\end{bmatrix}
\]  

(16)

Force laws that behave as \([s^1]\) or \([s^2]\) are the most promising. The acceleration increases for these laws as one scales down the system. The power that can be produced per unit volume also increases for these two laws. The surface tension scales advantageously, \([s^1]\), however, it is not clear how to use this force in most applications. Biological forces also scale well, \([s^2]\), but may be difficult to implement. Electrostatic and pressure-related forces appear to be useful forces in the small domain.
6. Conclusion

Watch makers have been the repository of fine mechanical skills, and their micro art is declining. Why then, should the micromechanical area start to grow now? The answer, in a word, is electronics: the same forces that are decreasing the usage of mechanical watches. Rapid progress has been made by the electronics industry using etching techniques to fabricate small mechanical parts. Once these micro structures are cut free from their silicon substrate and become independent components, the parts can be used to build a wide array of mechanical systems. The electronics industry is also placing increasingly severe demands on assembly tools. The future of assembly is with fast, accurate and gentle tooling. Electronics is already developing the micro sensors and intelligence needed to construct micro systems. This development of micro sensors and intelligence is important because, when a micro system breaks, it is difficult to use macro hands to find and fix the problem. What is now needed is the development of micro structures and actuators.

Other reasons for exploring the micro domain include the needs of medicine, space exploration and intellectual curiosity.

Given the desire to make micro systems, how does one start? Ten-centimeter sized systems can probably be made with modified macromachines: small electric motors, solenoids, encoders, etc. An interesting development in this size domain is robotic hands. Each finger is a small robot. Recently, hands have been developed using both magnetic electric motors and pneumatic actuators [38-41]. The centimeter- and millimeter-sized systems, however, will be very difficult to build with scaled-down versions of our macro equipment. They will probably require the development of new technologies. The following approach is suggested. First, build the necessary actuators and sensors, and then integrate these into systems.

As an existence proof and demonstration of the capabilities of small mechanical systems, nature proudly displays a wide array of small biological systems such as bacteria and cells.

Acknowledgements

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References

2. Small Machines, Large Opportunities: A Report on the Emerging Field of Microdynamics, A National Science Foundation report of the Workshop on Microelectromechanical Systems Research, 1980. (Copies of this report are available through G. A. Hazelrigg of the National Science Foundation, Washington, DC.)
Appendix

A. Introduction

In describing mechanical systems, one is interested in the behavior of the force laws. The scaling of current, and hence the magnetic force, will be described in Section B, and the scaling of the electrostatic force will be examined electromagnetically.

B. Scaling

This section deals with the scaling of forces per unit surface. The scaling of forces per unit area follows from the scaling of the electric field,

\[ \nabla \times \mathbf{B} = -c^2 \mathbf{E} \]

and the conservation of current:

\[ dF_b = I_b \cdot d \]

The electric field is characterized by the separation of variables, the 'd' loop, and

\[ \int (\nabla \times \mathbf{B}) \cdot d \leq 0 \]

or

\[ B_a = \frac{\mu_0}{2\pi d} \]

The nature of the cross product of the current and electric field is such that the scaling analysis results in non-linear behavior of the magnetic field.

Figure A1: Twisting and Torque

\[ I_a \]

\[ I_b \]

\[ d \]

\[ C \]

\[ \mathbf{B} \]

Fig. A1. Twisting and torque. The current and motor axes. The magnetic field is perpendicular to the plane of the paper.


examined in Section C. The equations used are Maxwell’s equations, the electromagnetic force law \( F = q(E + v \times B) \), and the equations of dynamics. The MKSA (SI) system of units is used throughout this Appendix.

B. Scaling of magnetic forces

This Section examines how forces that are generated magnetically scale. Three cases will be examined: constant current density, constant heat flow per unit surface area and constant temperature rise across the windings.

The scaling of the magnetic force can be calculated from the force law and one of Maxwell’s equations:

\[
dF_b = I_b \, dl_b \times B_a
\]  \hspace{1cm} (A1)

\[
\nabla \times B - c^{-2} \frac{\partial E}{\partial t} = \mu_0 J
\]  \hspace{1cm} (A2)

Figure A1 below shows a typical configuration. The \( F \) is the magnetic force, \( I \) is the total current through the wire, \( B \) is the magnetic induction, \( E \) is the electric field, \( t \) is the time, \( l \) measures the distance along the wires, and \( d \) is the separation between the wires. Integrating eqn. (A2) over the area of the ‘d’ loop, and using Stokes’s theorem, one has

\[
\int (\nabla \times B) \cdot dA = \int \left( c^{-2} \frac{\partial E}{\partial t} + \mu_0 J \right) \cdot dA = \int B \cdot dl = 2\pi d B = c^{-2} \int \frac{\partial E}{\partial t} \cdot dA + \mu_0 I_a
\]  \hspace{1cm} (A3)

or

\[
B_a = \frac{\mu_0}{2\pi d} I_a + \frac{1}{2\pi dc^2} \int \frac{\partial E}{\partial t} \cdot dA
\]  \hspace{1cm} (A4)

The \( B \) is equal to two terms, the current, \( I_a \), term, and an \( E \) term. The vector nature of the force equation, \( dF_b = I_b \, dl_b \times B_a \), and the values of the dot and cross products do not change as the system is scaled to smaller sizes, so we do not need to be concerned with the vector part of this equation in our scaling analysis. The example under consideration in Fig. A1 has been chosen so that the dot and cross products are equal to one. The force equation (A1) can now be written

![Diagram](image)

Fig. A1. Two wire segments of length \( l \) are shown separated by \( d \) and carrying currents \( I_a \) and \( I_b \). The forces on the wires are given by eqns. (A5) and (A6).
\[ dF_b = \frac{\mu_0}{2\pi d} I_a I_b \frac{l}{d} + \frac{1}{2\pi d c^2} I_b \int \frac{\partial E}{\partial t} \cdot \frac{dA}{dt} \ dA \]  

(A5)

Integrating along a length \( l \) of the wire, \( \int_0^l dF \),

\[ F_b = \frac{\mu_0}{2\pi} I_a I_b \frac{l}{d} + \frac{1}{2\pi d c^2} I_b \left[ \int \frac{\partial E}{\partial t} \cdot \frac{dA}{dt} \right] l \]  

(A6)

Now as all the dimensions of the system are scaled by the scale factor, \( s \), the first term scales as \( I^2 \). (When \( l \) doubles, so does \( d \), and their effect cancels.) It is more difficult to see how the second term scales. The first term is examined for three cases in the Subsections below: (1) constant current density; (2) constant heat flow through the surface of the wire, (3) constant temperature rise. These three assumptions lead to very different forces and power dissipations. The fourth Subsection discusses the forces resulting from the interaction of a permanent magnet and electromagnet. This force is given by eqn. (A1), \( dF_b = I_b \ dI_b \times B_a \). The fifth Subsection relates the second term, \( (\partial E/\partial t) \cdot dA \), to the three current cases.

**B.1. Constant current density**

Here the current density is assumed to be an intensive variable: its value does not change as the scale size is changed. In terms of our notation, \( J = [s^0] \). Figure A2 shows the end of one of the wires. The total current is

\[ I = \int J \cdot dA = JA \]  

(A7)

The area scales as \( A = [s^2] \), and

\[ I = [s^0] \cdot [s^2] = [s^2] \]  

(A8)

\[ F = \frac{\mu_0}{2\pi} I_a I_b \frac{l}{d} = [s^4] \]  

(A9)

The force scales as \( s^4 \). If the scale size decreases by a factor of 10, the force decreases by a factor of 10 000. This regime is quite unattractive.

**B.2. Constant heat flow through the surface of the wire**

In this case there are two intensive variables, the heat flow out of the wire per unit wire surface area (\( \dot{Q}/A_s \)), and the resistivity, \( \rho \), of the wire. The heat flow out of the wire must be equal to the power dissipated in the wire, \( \dot{Q} = P = I^2 R \). Figure A3 below shows a piece of the wire. The \( A_s \) is the area of the end of the wire, \( A_s \) is the surface area, and \( r \) is the radius. Now,

\[ \frac{dQ}{dt} = -kA \]

where \( Q \) is the area, \( T \)
Fig. A3. A piece of wire is resistively heated by the current flowing through the cross-section $A_e$, and cooled by the heat flowing through the surface area $A_s$.

\[ \frac{\dot{Q}}{A_s} = \rho = [s^0] \quad \text{(intensive variables)} \quad (A10) \]

\[ \dot{Q} = P = I^2 R = I^2 \left( \frac{\rho l}{A_e} \right) \quad (A11) \]

where the resistance equals the resistivity, $\rho$, times $(l/A_e)$. Scaling $\dot{Q}/A_s$ and using $A = [s^2]$,

\[ \frac{\dot{Q}}{A_s} = [s^0] = \frac{I^2 \rho l}{A_e A \rho} = I^2 [s^{-3}] \quad (A12) \]

\[ I = [s^{1.5}], \quad J = \frac{I}{A_e} = [s^{-0.5}] \quad (A13) \]

The force scales as $I^2$ and

\[ F = [s^3] \quad (A14) \]

For this case, when the scale decreases a factor of ten, the force decreases by a factor of 1000 and the current density increases by about a factor of three. As will be shown later, this increase in current density rapidly increases the resistive power dissipation.

**B.3. Constant temperature rise of the wire**

There is a maximum temperature that the wire and insulation can withstand. This Subsection assumes a maximum temperature rise between the coil and the surrounding ambient, independent of the scale size. The current density and the force scaling are calculated.

A section of a wire is shown in Fig. A4. The radius of the wire is $r_s$, the length of the section under consideration is $l$. For the integration an interior radius $r_1$ and an exterior radius $r_2$ are used. The equation of heat conduction is

\[ \frac{dQ}{dt} = -k A \frac{dT}{dx} \quad (A15) \]

where $Q$ is the heat flow, $t$ is the time, $k$ is the thermal conductivity, $A$ is the area, $T$ is the temperature and $x$ is distance. For a cylinder of radius $r_1$,
inside the wire, \(\frac{dQ}{dt}\) is equal to the resistive heat dissipated within this cylinder:

\[
\frac{dQ}{dt} = P = I^2R = [JA_{e1}]^2 \frac{\rho l}{A_{e1}} = J^2\rho A_{e1}l = J^2\rho \pi r_1^2 l \tag{A16}
\]

where \(R = \rho l/A_{e1}\) (\(\rho\) is the resistivity), \(A_{e1} = \pi r_1^2\) and \(I = JA_{e1}\). In eqn. (A15), \(A\) is the area of the cylinder surface, \(A_s = 2\pi r_1l\) and \(dT/dx\) becomes \(dT/dr\).

Rewriting eqn. (A15):

\[
\frac{1}{2} \int_{r=0}^{r_s} J^2\rho r_1 \, dr_1 = \frac{T_s}{T_0} - k \, dT \tag{A17}
\]

Integrating

\[
J^2\rho \left[ \frac{1}{4} \frac{r_s^2}{r_0^2} \right] = -k \left[ T_s - T_0 \right] = k \Delta T \tag{A18}
\]

The temperature rise in this case is intensive, \(\Delta T = [s^0]\), and scaling the above equation,

\[
J^2\rho [s^2] = [s^0] \tag{A19}
\]

\(J = [s^{-1}]\)

\(I = JA = [s^1]\)

\(F = [s^2]\)

Outside the wire, the total heat flux is constant, and eqn. (A15) becomes

\[
\frac{dQ}{dt} = P = J^2\rho \pi r_s^2 l \tag{A20}
\]

and the integral (A17) becomes

\[
\frac{1}{2} J^2\rho r_s^2 \int_{r_s}^{r_f} \frac{1}{r_2} \, dr_2 = \frac{T_f}{T_s} - k \, dT \tag{A21}
\]
Integrating
\[\frac{1}{2} J^2 \rho r_s^2 [\log_e r_t - \log_e r_s] = -k [T_t - T_s] = -k \Delta T \quad (A22)\]

\[\Delta T = -\frac{J^2 \rho r_s^2}{2k} \left[ \log_e \frac{r_t}{r_s} \right] \quad (A23)\]

Equation (A23) scales the same as eqn. (A18) except for the \(\log_e(r_t/r_s)\) factor. When the system is scaled down, the ratio of \(r_t\) to \(r_s\) must remain constant, and the \(\log_e\) term remains constant. (This says that the distance to the heat sink, \(r_t\), must scale.) Hence, (A23) scales exactly the same way as (A18), and the scaling laws for this case are given by (A19).

For this case of constant temperature rise, when the scale is decreased a factor of ten, the current density increases by a factor of ten, and the force decreases by a factor of one hundred.

\[B.4. A\ wire\ and\ permanent\ magnet\]

The force between a current-carrying wire and a permanent magnet is given by eqn. (A1)
\[dF_b = I_b \, dl_b \times B_a \quad (A1)\]

The scaling of \(I_b\) for the three cases has been derived in the previous Subsections and is given by \(s^2\) for the constant current density case, \(s^{1.5}\) for the constant heat flow case and \(s^1\) for the constant temperature rise case. The \(dl_b\) is a length and scales as \(s\), and the maximum \(B_a\) that can be conveniently produced depends upon the saturation field of the material used, and is an intensive variable \([s^0]\). Hence the force for these three cases scales as \(s^3\) for the constant current density, \(s^{2.5}\) for the constant heat flow and \(s^2\) for constant temperature rise. For comparison, the force between two current-carrying wires scales as: \(s^3\) for the constant current density, \(s^2\) for the constant heat flow, and \(s^2\) for constant temperature rise.

\[B.5. The\ J(dE/dt) \cdot dA\ term\ in\ the\ force\ equation\]

The second term in the force equation raises interesting possibilities. One hopes that this term leads to better scaling properties. Unfortunately, this term is just the displacement current, and as will be seen in the capacitor example below, this term is equivalent to the current term discussed above.

Equations (A1), (A3) and (A4) are reproduced below for reference.
\[dF_b = I_b \, dl_b \times B_a \quad (A1)\]
\[\int (\nabla \times B) \cdot dA = \int \left( c^{-2} \frac{\partial E}{\partial t} + \mu_0 J \right) \cdot dA = \int B \cdot dl = 2\pi dB = \mu_0 I_a + c^{-2} \int \frac{\partial E}{\partial t} \cdot dA \quad (A3)\]
or
\[ B_a = \frac{\mu_0}{2\pi d} I_a + \frac{1}{2\pi d c^2} \int \frac{\partial E}{\partial t} \cdot dA \] (A4)

The equality of these two terms can be shown for a capacitor by considering the integral of \( B \cdot d\ell \) around the plates of the capacitor. In Fig. A5, the large circle is the loop, and the smaller circles are the plates of the capacitor. This is equivalent to the surface integral of \( \nabla \times B \) over the area of the loop. If this surface cuts through one of the wires shown, then the first term incorporating \( I_a \) is appropriate, and since the \( E \) field is zero away from the capacitor plates, the second term is zero. If the surface passes through the capacitor plates, which are shown as the two disks, then there is zero current passing through the surface, and only the \( dE/dt \) term is non-zero. The plates of the capacitor are, of course, infinitely close, etc. The two terms are equivalent, and either produces the same \( B \) field.

![Diagram](image)

Fig. A5. The integral of \( B \cdot d\ell \) around the larger loop is equal to the integral of the time derivative of the \( E \) field, and the current density \( J \), as shown in eqn. (A3). When integrating around the capacitor plates, shown as small circles, either of the two terms can be made to vanish, and hence they must scale in the same way.

**C. Scaling of electric forces**

Electrostatic forces scale well into the small domain. Two cases are examined: first, where the electric field is constant, \([s^0]\); secondly, where the electric field increases slightly as the scale size is decreased, \([s^{-0.5}]\).

The electrostatic forces are calculated from the Lagrangian and the potential energy. The Lagrangian is

\[ \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0 \]

(LA24)

\[ L = T - U \]

If the potential energy is only a function of \( x \), \( U = U(x) \), and the kinetic energy is \( T = \frac{1}{2} m \dot{x}^2 \), then

\[ -\frac{\partial U}{\partial x} - \frac{d}{dt} m \dot{x} = 0 = -\frac{\partial U}{\partial x} - F \]

\[ F = -\frac{\partial U}{\partial x} \] (A25)
Considering A5, the large capacitor. This loop. If this incorporates the capacitor, the capacitor current passing the plates of which are equiv-

Fig. A6. The forces between the parallel plates of a capacitor of width \( w \), length \( l \) and separated by a distance \( d \) are given by eqn. (A25).

For a parallel plate capacitor, such as that shown in Fig. A6, the potential energy can be calculated as follows:

\[
U = \frac{1}{2} CV^2
\]

\[
C = \varepsilon_0 \frac{wl}{d} \quad V = Ed
\]

\[
U = \frac{1}{2} \varepsilon_0 wldE^2
\]

and explicitly,

\[
F = -\frac{1}{2} \varepsilon_0 \frac{\partial}{\partial x} [wldE^2]
\]

\[
F = [s^2]E^2
\]

Now,

\[
E = s^0 \quad \text{or} \quad s^{-0.5}
\]

and from eqn. (A28) above,

\[
F = s^2 \quad \text{or} \quad s^1
\]

The electrostatic force scales as \([s^2]\) or as \([s^1]\), assuming the \( E \) field scales as \([s^0]\) or \([s^{-0.5}]\). When the scale size is decreased by ten, the electrostatic force decreases by a factor of one hundred or ten.

(A24)

(A25)

(A26)

(A27)

(A28)

(A29)

(A30)